Magnetostatic Fields

According to Coulomb's law, a distribution of stationary charge produces a static electric field (electrostatic field). The analogous equation to Coulomb's law in electrostatics is the Biot-Savart law in magnetostatics. The Biot-Savart law shows that when charge moves at a constant rate (direct current - DC), a static magnetic field (magnetostatic field) is produced. When the rate of charge movement varies with time (for example, an alternating current - AC), we find that coupled electric and magnetic fields are produced (electromagnetic field).

Static magnetic fields are also produced by stationary permanent magnets. When permanent magnets are set in motion such that a time-varying magnetic field is produced, a time-varying electric field is simultaneously produced. A time-varying electric field cannot exist without a corresponding time-varying magnetic field and vice versa.

All of the previously defined equations for electrostatic fields have dual equations for magnetostatic fields where the magnetic field quantities have dual units to those of the corresponding electric field quantities. The so-called constitutive equations that relate the electric field to the electric flux density and the magnetic field to the magnetic flux density are

\[ \mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_o \mathbf{E} \]
\[ \mathbf{B} = \mu \mathbf{H} = \mu_r \mu_o \mathbf{H} \]

- **\( \mathbf{E} \)** - vector electric field (V/m)
- **\( \mathbf{D} \)** - vector electric flux density (C/m²)
- **\( \varepsilon \)** - total permittivity (F/m)
- **\( \varepsilon_r \)** - relative permittivity (unitless)
- **\( \varepsilon_o \)** - free space permittivity [8.854 \times 10^{-12} (F/m)]

- **\( \mathbf{H} \)** - vector magnetic field (A/m)
- **\( \mathbf{B} \)** - vector magnetic flux density (Wb/m² = T)
- **\( \mu \)** - total permeability (H/m)
- **\( \mu_r \)** - relative permeability (unitless)
- **\( \mu_o \)** - free space permeability [4π \times 10^{-7} (H/m)]

Wb - Weber
T - Tesla
Magnetic Forces on Current-Carrying Conductors

Given that charge moving in a magnetic field experiences a force, a current carrying conductor in a magnetic field also experiences a force. The current carrying conductor (modeled as a line current) can be subdivided into differential current elements (differential lengths). The charge-velocity product for a moving point charge can be related to an equivalent differential length of line current.

\[ q \, \mathbf{u} = q \, \frac{d\mathbf{l}'}{t} \, \hat{\mathbf{l}} = \frac{q}{t} \, d\mathbf{l}' \, \hat{\mathbf{l}} = I \hat{\mathbf{l}} d\mathbf{l}' = I d\mathbf{l}' \quad \text{(A - m)} \]

The equivalence of the moving point charge and the differential length of line current yields the equivalent magnetic force equation.

\[ F_m = q (\mathbf{u} \times \mathbf{B}) \quad \leftrightarrow \quad dF_m = d\mathbf{l}' \, (I \times \mathbf{B}) \]

The total force on the current carrying conductor is found by summing the forces on all of the differentials elements of current (integrating along the length of the conductor).

\[ F_m = \int_L (I \times \mathbf{B}) d\mathbf{l}' \]

Given a steady current, the magnitude of the current is constant along the length of the conductor so that the magnetic force can be written as

\[ F_m = I \int_L (d\mathbf{l}' \times \mathbf{B}) \]
Torque on a Current Loop

Given the change in current directions around a closed current loop, the magnetic forces on different portions of the loop vary in direction. Using the Lorentz force equation, we can show that the net force on a simple circular or rectangular loop is a torque which forces the loop to align its magnetic moment with the applied magnetic field.

Consider the rectangular current loop shown below. The loop lies in the $x$-$y$ plane and carries a DC current $I$. The loop lies in a uniform magnetic flux density $B$ given by

$$B = B_y \hat{y} + B_z \hat{z}$$

The loop consists of four distinct vector current segments.

$$I_1 = I \hat{y} \quad I_2 = -I \hat{x} \quad I_3 = -I \hat{y} \quad I_4 = I \hat{x}$$

Given a uniform flux density and a DC current along straight current segments, the magnetic force on each conductor segment can be simplified to the following equation.

$$F_m = \int_L (I \times B) dl'$$

$$= (I \times B) \int_L dl'$$

$$= (I \times B) L$$

The forces on the current segments can be determined for each component of the magnetic flux density.
Forces due to $B_z$ (net force = 0)

\[ F_1 = (I \hat{y} \times B_z \hat{z}) l_1 = IB_z l_1 \hat{x} \]
\[ F_2 = (-I \hat{x} \times B_z \hat{z}) l_2 = IB_z l_2 \hat{y} \]
\[ F_3 = (-I \hat{y} \times B_z \hat{z}) l_1 = -IB_z l_1 \hat{x} \]
\[ F_4 = (I \hat{x} \times B_z \hat{z}) l_2 = -IB_z l_2 \hat{y} \]

Forces due to $B_y$ (net force = vector torque $T$)

\[ F_1 = (I \hat{y} \times B_y \hat{y}) l_1 = 0 \]
\[ F_2 = (-I \hat{x} \times B_y \hat{y}) l_2 = -IB_y l_2 \hat{z} \]
\[ F_3 = (-I \hat{y} \times B_y \hat{y}) l_1 = 0 \]
\[ F_4 = (I \hat{x} \times B_y \hat{y}) l_2 = IB_y l_2 \hat{z} \]

The vector torque on the loop is defined in terms of the force magnitude ($IB_y l_2$), the torque moment arm distance ($l_1/2$), and the torque direction (defined by the right hand rule):

\[ T = 2(IB_y l_2) \left( \frac{l_1}{2} \right) (-\hat{x}) = -IB_y l_1 l_2 \hat{x} = -IA B_y \hat{x} \]

where $A = l_1 l_2$ is the loop area. The vector torque can be written in a general way in terms of the vector magnetic moment ($m$) of the loop.

\[ m = IA \hat{n} \quad \text{(vector magnetic moment)} \]

where $\hat{n}$ is the unit normal to the loop (defined by the right hand rule as applied to the current direction). The vector torque in terms of the magnetic moment is

\[ T = m \times B = m B \sin \theta \hat{a}_T = IA B \sin \theta \hat{a}_T \]

Note that the torque on the loop tends to align the loop magnetic moment with the direction of the applied magnetic field.
Biot-Savart Law

The Biot-Savart law defines the magnetostatic field produced by a steady current. The overall magnetic field produced by an arbitrary vector line current \( I \) (filament, zero cross-section) is expressed as a line integral of the current. According to the Biot-Savart law, the differential vector magnetic field \( d\mathbf{H} \) at the field point \( P \) produced by a differential element of current \( Idl' \) is

\[
d\mathbf{H} = \frac{1}{4\pi} \frac{I \times \hat{a}_{R_o}}{R_o^2} dl'
\]

\[
R_o = |\mathbf{R} - \mathbf{R}'|
\]

\[
\hat{a}_{R_o} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}
\]

\[
dl' = dl' \hat{\mathbf{i}}
\]

\[
d\mathbf{H} = \frac{I \sin \alpha}{4 \pi R_o^2} dl'
\]

Note that the direction of the magnetic field is given by the direction of \( I \times (\mathbf{R} - \mathbf{R}') \). The overall vector magnetic field produced by the line current at the field point \( P \) is found by integrating \( d\mathbf{H} \) along the length of the line current which yields

\[
\mathbf{H} = \int_L d\mathbf{H} = \frac{1}{4\pi} \int_L \frac{I \times \hat{a}_{R_o}}{R_o^2} dl' = \frac{1}{4\pi} \int_L \frac{I \times (\mathbf{R} - \mathbf{R}')}{{R_o}^3} dl'
\]

\[
\text{line current}
\]

For surface currents \( J_s, \text{A/m} \) and volume currents \( J, \text{A/m}^2 \), the units on the differential elements of current are equivalent to that of the line current:

\[
Idl' \leftrightarrow J_s ds' \leftrightarrow J dv' \quad \text{(A-m)}
\]

Inserting the equivalent element of current into the equation for \( d\mathbf{H} \) and integrating over the total current distribution yields the equations for the magnetic fields produced by surface and volume currents.

\[
\mathbf{H} = \frac{1}{4\pi} \int_S \frac{J_s \times \hat{a}_{R_o}}{R_o^2} ds' = \frac{1}{4\pi} \int_S \frac{J_s \times (\mathbf{R} - \mathbf{R}')}{{R_o}^3} ds'
\]

\[
\text{surface current}
\]
\[ H = \frac{1}{4\pi} \iiint_V \frac{J \times \hat{a}_{R_o}}{R_o^2} \, dv' = \frac{1}{4\pi} \iiint_V \frac{J \times (R - R')}{R_o^3} \, dv' \] (volume current)

Example (Biot-Savart law / line current)

Determine the magnetic field of a line segment of current lying along the z-axis extending from \( z = z_A \) to \( z = z_B \).

\[ H = \frac{1}{4\pi} \int_L \frac{I \times (R - R')}{R_o^3} \, dl' \]
\[ I = I \hat{z} \quad dl' = dz' \]

\[ R = r \hat{r} + z \hat{z} \]
\[ R' = z' \hat{z} \]
\[ R - R' = r \hat{r} + (z - z') \hat{z} \]
\[ R_o = \sqrt{r^2 + (z - z')^2} \]

\[ H = \frac{1}{4\pi} \int_{z_A}^{z_B} \frac{(I \hat{z}) \times [r \hat{r} + (z - z') \hat{z}]}{[r^2 + (z - z')^2]^{3/2}} \, dz' \]

\[ = \frac{1}{4\pi} \int_{z_A}^{z_B} \frac{Ir \hat{\phi}}{[r^2 + (z - z')^2]^{3/2}} \, dz' \]

\[ = \frac{Ir \hat{\phi}}{4\pi} \int_{z_A}^{z_B} \frac{dz'}{[r^2 + (z - z')^2]^{3/2}} \]

(Given the field point \( P \), the direction of \( \hat{\phi} \) does not change as \( z' \) is varied along the length of the current)

Transformation of variable: \[ \alpha = z - z' \]
\[ d\alpha = -dz' \]

\[ z' = z_A \quad \Rightarrow \quad \alpha = z - z_A \]
\[ z' = z_B \quad \Rightarrow \quad \alpha = z - z_B \]
\[ H = \frac{I r \hat{\phi}}{4\pi} \int_{z-z_A}^{z-z_B} \frac{-d\alpha}{(\alpha^2+r^2)^{3/2}} = -\frac{I r \hat{\phi}}{4\pi} \left[ \frac{\alpha}{r^2(\alpha^2+r^2)^{1/2}} \right]_{z-z_A}^{z-z_B} \]

\[ = -\frac{I}{4\pi r} \left[ \frac{z-z_B}{\left[ r^2 + (z-z_B)^2 \right]^{1/2}} - \frac{z-z_A}{\left[ r^2 + (z-z_A)^2 \right]^{1/2}} \right] \hat{\phi} \]

\[ = \frac{I}{4\pi r} \left[ \frac{z-z_A}{R_A} - \frac{z-z_B}{R_B} \right] \hat{\phi} \]

Given this general result for the magnetic field of a current segment, we may apply it to several special cases.

(1)  Line current, symmetric about the x-y plane, \( z_A = -z_o, z_B = z_o \)

\[ H = \frac{I}{4\pi r} \left[ \frac{z+z_o}{R_A} - \frac{z-z_o}{R_B} \right] \hat{\phi} \]

If the field point lies in the x-y plane, then \( z = 0, R_A = R_B = \left[ r^2 + z_o^2 \right]^{1/2} \)

\[ H = \frac{I}{4\pi r} \left[ \frac{2z_o}{\sqrt{r^2 + z_o^2}} \right] \hat{\phi} = \frac{Iz_o}{2\pi r \sqrt{r^2 + z_o^2}} \hat{\phi} \]

(2)  Semi-infinite line current, \( z \in (0, \infty), z_A = 0, z_B = \infty, \]
\( R_A = \left[ r^2 + z^2 \right]^{1/2}, R_B = \infty \)

\[ \lim_{z_B \to -\infty} \frac{z-z_B}{R_B} = \lim_{z_B \to -\infty} \frac{z-z_B}{\left[ r^2 + (z-z_B)^2 \right]^{1/2}} = \lim_{z_B \to -\infty} \left[ \frac{r^2 + (z-z_B)^2}{z_B^2} \right]^{1/2} = \frac{z}{z_B} - 1 \]
\[ H = \frac{I}{4\pi r} \left[ 1 + \frac{z}{\sqrt{r^2 + z^2}} \right] \hat{\phi} \]

If the field point lies in the x-y plane \((z = 0)\),

\[ H = \frac{I}{4\pi r} \hat{\phi} \]

(3) Infinite line current, \(z' \in (-\infty, \infty)\)
Wherever the field point is located, the infinite length line current can be viewed as two semi-infinite line currents. The resulting magnetic field is twice that of the semi-infinite length current segment.

\[ H = 2 \frac{I}{4\pi r} \hat{\phi} = \frac{I}{2\pi r} \hat{\phi} \]

The previous formulas are useful when determining the magnetic field of a closed current loop made up of straight segments. The principle of superposition may be applied to determine the total magnetic field produced by the loop. The total magnetic field produced by the loop is the vector sum of the magnetic field contributions from each current segment.
Magnetic Field Due to a Circular Current Loop

The Biot-Savart law can be used to determine the magnetic field at the center of a circular loop of steady current.

\[ H = \frac{1}{4\pi} \int_L \frac{I \times (R - R')}{R_o^3} \, dl' \]

\[ I = I\hat{\phi} \]

\[ dl' = ad\phi' \]

\[ R = h\hat{z} \]

\[ R' = a\hat{r} \]

\[ R - R' = h\hat{z} - a\hat{r} \]

\[ R_o = |R - R'| = \sqrt{a^2 + h^2} \]

Symmetry of the loop magnetic field when the field point lies on the loop axis
\[ H = \frac{1}{4\pi} \int_0^{2\pi} \frac{\left( I \dot{\phi} \right) \times (h \hat{z} - a \hat{r})}{(a^2 + h^2)^{3/2}} \, a d\phi' \]

\[ = \frac{I}{4\pi} \int_0^{2\pi} \frac{h \hat{r} + a \hat{z}}{(a^2 + h^2)^{3/2}} \, a d\phi' \]

\[ \hat{r} = \cos \phi' \hat{x} + \sin \phi' \hat{y} \]

\[ H = \frac{Ia}{4\pi (a^2 + h^2)^{3/2}} \left[ h \hat{x} \int_0^{2\pi} \cos \phi' \, d\phi' + h \hat{y} \int_0^{2\pi} \sin \phi' \, d\phi' + a \hat{z} \int_0^{2\pi} \, d\phi' \right] \]

The horizontal components of the magnetic field integrate to zero as was shown previously, by symmetry.

\[ H = \frac{Ia}{4\pi (a^2 + h^2)^{3/2}} \left[ 2\pi a \hat{z} \right] \]

\[ = \frac{Ia^2}{2(a^2 + h^2)^{3/2}} \hat{z} \]

At the loop center \( (h=0) \),

\[ H = \frac{I}{2a} \hat{z} \]
Forces Between Current Carrying Conductors

Given that any current carrying conductor produces a magnetic field, when two current carrying conductors are brought into close proximity, each conductor lies in the magnetic field of the other conductor. Therefore, both current carrying conductors exert a force on the opposite conductor.

Example (Force between line currents)

Determine the force/unit length on a line current $I_1$ due to the magnetic flux produced by a parallel line current $I_2$ (separation distance = $d$) flowing in the opposite direction.

The magnetic flux density at the location of $I_1$ due to the current $I_2$ is

$$B_2(x=0, y=0) = \frac{\mu I_2}{2\pi d} (-\hat{x})$$

The force on a length $l$ of current $I_1$ due to the flux produced by $I_2$ is

$$F_1 = \int_0^l (I_1 \times B_2) \, dz'$$

$$= \int_0^l \left[ (I_1 \hat{z}) \times \left( \frac{\mu I_2}{2\pi d} (-\hat{x}) \right) \right] \, dz'$$

$$= \frac{\mu I_1 I_2}{2\pi d} (-\hat{y}) \int_0^l dz'$$

$$= \frac{-\mu I_1 I_2 l}{2\pi d} \hat{y}$$
The force per unit length on the current \( I_1 \) is

\[
\frac{F_1}{l} = -\frac{\mu I_1 I_2}{2 \pi d'} \hat{y}
\]

The force on a length \( l \) of current \( I_2 \) due to the flux produced by \( I_1 \) is

\[
F_2 = \int_0^l (I_2 \times B_1) \, dz'
\]

\[
= \int_0^l \left[ (-I_2 \hat{z}) \times \left( \frac{\mu I_1}{2 \pi d'} (-\hat{x}) \right) \right] \, dz'
\]

\[
= \frac{\mu I_1 I_2}{2 \pi d'} (\hat{y}) \int_0^l dz'
\]

\[
= \frac{\mu I_1 I_2 l}{2 \pi d'} \hat{y}
\]

The force per unit length on the current \( I_2 \) is

\[
\frac{F_2}{l} = \frac{\mu I_1 I_2}{2 \pi d'} \hat{y}
\]

Note that the currents repel each other given the currents flowing in opposite directions. If the currents flow in the same direction, they attract each another.
Ampere's Law

Gauss's law is the Maxwell equation that relates the electrostatic field to the source of the electrostatic field (charge). Ampere's law is the Maxwell equation that relates the magnetostatic field to the source of the magnetostatic field (current).

\[ \oint S D \cdot ds = Q \quad \text{(Gauss's law – integral form)} \]

\[ \oint L H \cdot dl = I \quad \text{(Ampere's law – integral form)} \]

**Ampere's Law** - The line integral of the magnetic field around a closed path equals the net current enclosed (the current direction is implied by the direction of the path according to the right hand rule).

**Example** (Ampere’s law / infinite-length line current)

Given a infinite-length line current \( I \) lying along the \( z \)-axis, use Ampere’s law to determine the magnetic field by integrating the magnetic field around a circular path of radius \( r \) lying in the \( x-y \) plane.

From Ampere’s law,

\[ \oint L H \cdot dl = \oint L H_\phi \cdot dl = I \]

By symmetry, the magnetic field is uniform on the given path so that

\[ H_\phi \oint L dl = H_\phi (2 \pi r) = I \]

or

\[ H_\phi = \frac{I}{2 \pi r} \]

This result agrees with that found using the Biot-Savart law.
Example  (Ampere’s law / Magnetic field in a coaxial transmission line)

The coaxial transmission line shown below carries a total current $I$ in the $+a_z$ direction through the inner conductor and in the $-a_z$ direction through the outer conductor. Assume uniform current densities in both conductors of the coaxial transmission line. Use Ampere’s law to determine the magnetic field everywhere.

![Diagram of coaxial transmission line]

The uniform vector current densities in the inner and outer conductors of the coaxial transmission line ($J_i$ and $J_o$, respectively) are

- **inner conductor**
  \[ J_i = \frac{I}{A_i} \hat{z} = \frac{I}{\pi a^2} \hat{z} = J_i \hat{z} \]

- **outer conductor**
  \[ J_o = \frac{I}{A_o} (-\hat{z}) = -\frac{I}{\pi (c^2 - b^2)} \hat{z} = -J_o \hat{z} \]

To determine the magnetic field everywhere, Ampere’s law is applied on circular paths in the four distinct regions for the coaxial transmission line.
\[ L_1 \quad (r < a) \quad [\text{region within the inner conductor}] \]
\[ L_2 \quad (a < r < b) \quad [\text{region between the conductors}] \]
\[ L_3 \quad (b < r < c) \quad [\text{region within the outer conductor}] \]
\[ L_4 \quad (r > c) \quad [\text{region outside the outer conductor}] \]

By symmetry, on all four of the integration paths, the magnetic field is uniform and has only a \( \hat{\phi} \) component. Thus, for each path, Ampere’s law reduces to

\[
\oint H \cdot dl = \oint H_{\hat{\phi}} dl = H_{\hat{\phi}} \oint dl = H_{\hat{\phi}} (2\pi r) = I_{\text{enclosed}}
\]

or

\[
H_{\hat{\phi}} = \frac{I_{\text{enclosed}}}{2\pi r} \quad \quad H = \frac{I_{\text{enclosed}}}{2\pi r} \hat{\phi}
\]

The magnetic field in each region is proportional to the net current enclosed by the path.

**Within the inner conductor \((r < a)\) \([L_1]\)**

\[
I_{\text{enclosed}} = \iint_S J \cdot ds = \iint_S (J_i \hat{z}) \cdot (ds \hat{z}) = J_i \iint_S ds = J_i (\pi r^2)
\]

\[
= \frac{I}{\pi a^2} (\pi r^2) = I \frac{r^2}{a^2}
\]

\[ H = \frac{I_{\text{enclosed}}}{2\pi r} \hat{\phi} = \frac{I r^2}{2\pi r a^2} \hat{\phi} = \frac{I r}{2\pi a^2} \hat{\phi} \quad (r < a)
\]

**Between the conductors \((a < r < b)\) \([L_2]\)**

\[ I_{\text{enclosed}} = I \]

\[ H = \frac{I_{\text{enclosed}}}{2\pi r} \hat{\phi} = \frac{I}{2\pi r} \hat{\phi} \quad (a < r < b) \]
Within the outer conductor \((b < r < c)\) \([L_3]\)

\[
I_{\text{enclosed}} = I + \oint_{S_o} \mathbf{J} \cdot d\mathbf{s} = I + \oint_{S_o} (-J_0 \hat{z}) \cdot (d\mathbf{s} \hat{z}) = I - J_0 \oint_{S_o} d\mathbf{s}
\]

\[
= I - \left[ \frac{I}{\pi (c^2 - b^2)} \right] \left[ \pi (r^2 - b^2) \right] = I \left[ 1 - \frac{r^2 - b^2}{c^2 - b^2} \right]
\]

\[
\mathbf{H} = \frac{I_{\text{enclosed}}}{2 \pi r} \hat{\phi} = \frac{I}{2 \pi r} \left[ 1 - \frac{r^2 - b^2}{c^2 - b^2} \right] \hat{\phi} \quad (b < r < c)
\]

Outside the outer conductor \((r > c)\) \([L_4]\)

\[
I_{\text{enclosed}} = I + (-I) = 0
\]

\[
\mathbf{H} = \frac{I_{\text{enclosed}}}{2 \pi r} \hat{\phi} = 0 \quad (r > c)
\]

The magnetic field of a single conductor carrying a uniform current density can be determined from the results of the coaxial transmission line. With no outer conductor, the curve for the region between the coaxial conductors would continue for the single conductor.
Example (Ampere’s law / magnetic field of a surface current)

Determine the vector magnetic field produced by a uniform surface current covering the $x$-$y$ plane flowing in the $\hat{x}$ direction.

\[
\mathbf{J}_s = J_o \hat{x}
\]

Ampere’s law may be applied on the closed path shown above such that

\[
\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{\mathbf{1}} \mathbf{H} \cdot d\mathbf{l} + \int_{\mathbf{2}} \mathbf{H} \cdot d\mathbf{l} + \int_{\mathbf{3}} \mathbf{H} \cdot d\mathbf{l} + \int_{\mathbf{4}} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}}
\]

To evaluate the Ampere’s law integral, we must first determine the vector characteristics of the magnetic field. By symmetry, the magnetic field on the horizontal segments of the path (2 and 4) must be uniform. Also by symmetry, we can show that the magnetic field above the surface current is everywhere $-\hat{y}$ directed while the magnetic field everywhere below the surface current is $+\hat{y}$ directed. The overall surface current can be subdivided into differential lengths ($J_o \, dy$) with each differential length equivalent to a line current. For any given field point, and any given line current, there is always another line current in the opposite direction that produces a magnetic field component that when added to the magnetic field component of the original line current, produces only a $-\hat{y}$ component (above) or a $+\hat{y}$ component (below) of magnetic field.
With only horizontal components of magnetic field, the Ampere’s law integrals on the vertical paths (1 and 3) are zero. The magnetic field on the horizontal paths (2 and 4) may be written as

\[
H = \begin{cases} 
-H_o \hat{y} & (z > 0) \\
H_o \hat{y} & (z < 0) 
\end{cases}
\]

where \(H_o\) is a constant (the magnetic field is uniform on the paths). Given the magnetic field characteristics on the horizontal and vertical paths, the Ampere’s law integral can be evaluated to determine the magnetic field of the surface current.

\[
\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{-w}^{+w} \mathbf{H} \cdot d\mathbf{l} + \int_{-w}^{+w} \mathbf{H} \cdot d\mathbf{l} \\
= \int_{-w}^{+w} (-H_o \hat{y}) \cdot (-dy)(-\hat{y}) + \int_{-w}^{+w} (H_o \hat{y}) \cdot (dy)(\hat{y}) \\
= -H_o \int_{-w}^{+w} dy + H_o \int_{-w}^{+w} dy = -H_o (-2w) + H_o (2w) = H_o (4w)
\]

According to Ampere’s law, this integral is equal to the total current enclosed by the path (flowing in the \(+\hat{x}\) direction for the given path). The total current enclosed is the surface current located between \(x=-w\) and \(x=+w\). For a uniform surface current, the total current is the product of the surface current density and the distance (2w) such that \(I_{enclosed} = J_o (2w)\).

\[
\oint_L \mathbf{H} \cdot d\mathbf{l} = H_o (4w) = I_{enclosed} = J_o (2w) \quad \Rightarrow \quad H_o = \frac{J_o}{2}
\]

The magnetic field above and below the current sheet is

\[
H = \begin{cases} 
\frac{-J_o}{2} \hat{y} & (z > 0) \\
\frac{J_o}{2} \hat{y} & (z < 0) 
\end{cases}
\]

Note that the magnetic field is uniform everywhere above and below the surface current.
Differential Form of Ampere’s Law
(Curl Operator)

The differential form of Ampere’s law relates the vector magnetic field \( \mathbf{H} \) to the vector current density \( \mathbf{J} \) at a point. The three rectangular coordinate current density components \( (J_x, J_y, J_z) \) at the point \( P \) shown below can be related to the magnetic field by applying Ampere’s law to the three differential surfaces \( (\Delta S_x, \Delta S_y, \text{ and } \Delta S_z) \) which are bounded by paths \( (L_x, L_y, \text{ and } L_z) \). Dividing the Ampere’s law integrals by the differential surfaces and taking the limit as each differential surface approaches zero defines the \textit{curl} of the magnetic field.

\[
\text{curl}\, \mathbf{H} = \lim_{\Delta S_x \to 0} \frac{\oint_{L_x} \mathbf{H} \cdot d\mathbf{l}}{\Delta S_x} \hat{x} + \lim_{\Delta S_y \to 0} \frac{\oint_{L_y} \mathbf{H} \cdot d\mathbf{l}}{\Delta S_y} \hat{y} + \lim_{\Delta S_z \to 0} \frac{\oint_{L_z} \mathbf{H} \cdot d\mathbf{l}}{\Delta S_z} \hat{z}
\]

\[
= J_x \hat{x} + J_y \hat{y} + J_z \hat{z} = \mathbf{J}
\]

Thus, the differential form of Ampere’s law is

\[
\text{curl}\, \mathbf{H} = \mathbf{J}
\]

The curl operator in rectangular coordinates is

\[
\text{curl}\, \mathbf{H} = \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \hat{x} + \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \hat{y} + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \hat{z}
\]

\[
= J_x \mathbf{a}_x + J_y \mathbf{a}_y + J_z \mathbf{a}_z = \mathbf{J}
\]
The curl operator can be written in terms of the gradient operator as

$$\text{curl} \mathbf{H} = \nabla \times \mathbf{H} = \left( \frac{\partial}{\partial x} \mathbf{\hat{x}} + \frac{\partial}{\partial y} \mathbf{\hat{y}} + \frac{\partial}{\partial z} \mathbf{\hat{z}} \right) \times \left[ H_x \mathbf{\hat{x}} + H_y \mathbf{\hat{y}} + H_z \mathbf{\hat{z}} \right]$$

which can also be written in determinant form as

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{\hat{x}} & \mathbf{\hat{y}} & \mathbf{\hat{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

The same technique is used to determine the curl operator in cylindrical and spherical coordinates. In these cases, we must use differential surfaces that match the given coordinate system.

$$\nabla \times \mathbf{H} = \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{\hat{x}} + \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{\hat{y}} + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{\hat{z}}$$

(rectangular)

$$= \left( \frac{1}{r} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \mathbf{\hat{r}} + \left( \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) \mathbf{\hat{\phi}}$$

$$+ \frac{1}{r} \left( \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial H_r}{\partial \phi} \right) \mathbf{\hat{z}}$$

(cylindrical)

$$= \frac{1}{R \sin \theta} \left( \frac{\partial}{\partial \theta} \left( H_\phi \sin \theta \right) - \frac{\partial H_\theta}{\partial \phi} \right) \mathbf{\hat{R}}$$

$$+ \frac{1}{R} \left( \frac{1}{\sin \theta} \frac{\partial H_R}{\partial \phi} - \frac{\partial}{\partial R} \left( R H_\phi \right) \right) \mathbf{\hat{\theta}}$$

$$+ \frac{1}{R} \left( \frac{\partial}{\partial R} \left( R H_\theta \right) - \frac{\partial H_R}{\partial \theta} \right) \mathbf{\hat{\phi}}$$

(spherical)
Example (Differential form of Ampere’s law)

Given the magnetic field inside and outside a conductor of radius $a$ carrying a uniform current density (total current = $I_{\text{out}}$), show that the differential form of Ampere’s law yields the current density in both regions.

\[
H_i = \frac{I}{2\pi a^2} \hat{\phi} \quad (r < a)
\]

\[
H_o = \frac{I}{2\pi r} \hat{\phi} \quad (r > a)
\]

From the differential form of Ampere’s law, evaluating the curl (in cylindrical coordinates) of the magnetic field inside and outside the conductor should yield the current density inside and outside the conductor. The current density inside and outside the conductor is

\[
J_i = \frac{I}{\pi a^2} \hat{z} \quad (r < a)
\]

\[
J_o = 0 \quad (r > a)
\]

The curl of the magnetic field in cylindrical coordinates is

\[
\nabla \times H = \left( \frac{1}{r} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \hat{\phi} + \left( \frac{\partial H_\phi}{\partial z} - \frac{\partial H_z}{\partial r} \right) \hat{\phi}
\]

\[
+ \frac{1}{r} \left( \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial H_r}{\partial \phi} \right) \hat{z}
\]

Since the magnetic fields for $r < a$ and $r > a$ contain only $\hat{\phi}$ components that are functions of $r$ only, all terms in the curl expression are zero except the first term of the $\hat{z}$ component.
Application of the curl operator yields

$$\nabla \times \mathbf{H} = \frac{1}{r} \left( \frac{\partial}{\partial r} \left( r H_\phi \right) \right) \hat{z}$$

$$= \begin{cases} 
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{Ir^2}{2\pi a^2} \right) \hat{z} & (r < a) \\
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{I}{2\pi} \right) \hat{z} & (r > a) 
\end{cases}$$

$$= \begin{cases} 
\frac{I}{r \left( 2\pi a^2 \right)} (2r) \hat{z} & (r < a) \\
\frac{1}{r} (0) \hat{z} & (r > a) 
\end{cases}$$

$$= \begin{cases} 
\frac{I}{\pi a^2} \hat{z} = \mathbf{J}_i & (r < a) \\
0 \hat{z} = \mathbf{J}_o & (r > a) 
\end{cases}$$

**Stoke’s Theorem**

*Stoke’s theorem* is a vector identity that defines the transformation of a line integral of a vector around a closed path into a surface integral over the surface bounded by that path. The integrand of the resulting surface integral is the curl of the vector. The general form of Stoke’s theorem is

$$\oint_L \mathbf{F} \cdot dl = \iint_S (\nabla \times \mathbf{F}) \cdot ds$$  \hspace{1cm} (Stoke’s theorem)

where the closed path $L$ defines the periphery of the surface $S$. As always, the path direction $dl$ and the surface normal $ds$ are related by the right hand rule. We can apply Stoke’s theorem to transform the integral form of Ampere’s law into its differential form.
The integral form of Ampere’s law is
\[ \oint_L \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}} = \iint_S \mathbf{J} \cdot d\mathbf{s} \]

Applying Stoke’s theorem allows the line integral of the magnetic field to be transformed into a surface integral so that
\[ \oint_L \mathbf{H} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \iint_S \mathbf{J} \cdot d\mathbf{s} \]
Since the surface integrals in this equation are valid for any surface \( S \), the integrands of the two integrals must be equal. This yields Ampere’s law in differential form for magnetostatic fields: \( \nabla \times \mathbf{H} = \mathbf{J} \).

**Gauss’s Law for Magnetic Fields**

In addition to Ampere’s law, the magnetostatic field is governed by *Gauss’s law for magnetic fields*. The integral and differential forms of Gauss’s law for magnetic fields are identical to those for electric fields except for the source terms (electric charge). To form Gauss’s law for magnetic fields, we first replace the electric flux density \( \mathbf{D} \) terms in Gauss’s law for electric fields by the magnetic flux density \( \mathbf{B} \). The electric charge terms in Gauss’s law for electric fields are replaced by zero since the dual parameter to electric charge (magnetic charge) does not exist.

**Gauss’s Law – electric fields**

(\textit{integral form})
\[ \iiint_S \mathbf{D} \cdot d\mathbf{s} = Q_{\text{enclosed}} \]

**Gauss’s Law – magnetic fields**

(\textit{integral form})
\[ \iiint_S \mathbf{B} \cdot d\mathbf{s} = 0 \]

**Gauss’s Law – electric fields**

(\textit{differential form})
\[ \nabla \cdot \mathbf{D} = \rho_v \]

**Gauss’s Law – magnetic fields**

(\textit{differential form})
\[ \nabla \cdot \mathbf{B} = 0 \]

The characteristics of electrostatic and magnetostatic fields are fundamentally different based on the existence of electric charge and the nonexistence of magnetic charge.
Electrostatic Fields

Electric flux lines begin on positive charge and end on negative charge.
(Discontinuous)

Magnetostatic Fields

Magnetic flux lines form closed loops
(Continuous)

Magnetization

Just as dielectric materials are polarized under the influence of an applied electric field, certain materials can be magnetized under the influence of an applied magnetic field. Magnetization for magnetic fields is the dual process to polarization for electric fields. The magnetization process may be defined using the magnetic moments of the electron orbits within the atoms of the material. Each orbiting electron can be viewed as a small current loop with an associated magnetic moment.

An unmagnetized material can be characterized by a random distribution of the magnetic moments associated with the electron orbits. These randomly oriented magnetic moments produce magnetic field components that tend to cancel one another (net $H=0$). Under the influence of an applied magnetic field, many of the current loops align their magnetic moments in the direction of the applied magnetic field.

Unmagnetized (random moments)  Magnetized (aligned moments)
If most of the magnetic moments stay aligned after the applied magnetic field is removed, a *permanent magnet* is formed.

A small current loop is commonly referred to as a *magnetic dipole*. The far field electric field of the electric dipole (produced during polarization) is functionally the same as the magnetic field of the magnetic dipole (produced during magnetization).

$$E = \frac{p}{4 \pi \varepsilon r^3} \left[ 2 \cos \theta a_r + \sin \theta a_\theta \right] \quad \text{(electric dipole)}$$

$$B = \frac{\mu m}{4 \pi r^3} \left[ 2 \cos \theta a_r + \sin \theta a_\theta \right] \quad \text{(magnetic dipole)}$$

The preceding equations assume the dipole is centered at the coordinate origin and oriented with its dipole moment along the $z$-axis.

The parameters associated with the magnetization process are duals to those of the polarization process. The magnetization vector $M$ is the dual of the polarization vector $P$ and is defined as the magnetic dipole moment per unit volume.

<table>
<thead>
<tr>
<th>Magnetization</th>
<th>Polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = \frac{m}{v} \left( \frac{\text{magnetic moment}}{\text{unit volume}} \right)$</td>
<td>$P = \frac{q l}{v} \left( \frac{\text{dipole moment}}{\text{unit volume}} \right)$</td>
</tr>
<tr>
<td>$B = \mu_o (H + M) = \mu_o \mu_r H = \mu H$</td>
<td>$D = \varepsilon_o E + P = \varepsilon_o \varepsilon_r E = \varepsilon E$</td>
</tr>
<tr>
<td>$\mu_r = 1 + \frac{M}{H} = 1 + \chi_m$</td>
<td>$\varepsilon_r = 1 + \frac{P}{\varepsilon_o E} = 1 + \chi_e$</td>
</tr>
<tr>
<td>$M = \chi_m H$</td>
<td>$P = \chi_e \varepsilon_o E$</td>
</tr>
</tbody>
</table>

Note that the magnetic susceptibility $\chi_m$ is defined somewhat differently than the electric susceptibility $\chi_e$. However, just as the electric susceptibility and relative permittivity are a measure of how much polarization occurs in the material, the magnetic susceptibility and relative permeability are a measure of how much magnetization occurs in the material.
Magnetic Materials

Magnetic materials can be classified based on the magnitude of the relative permeability. Materials with a relative permeability of just under one (a small negative magnetic susceptibility) are defined as diamagnetic. In diamagnetic materials, the magnetic moments due to electron orbits and electron spin are very nearly equal and opposite such that they cancel each other. Thus, in diamagnetic materials, the response to an applied magnetic field is a slight magnetic field in the opposite direction. Superconductors exhibit perfect diamagnetism ($\chi_m = -1$) at temperatures near absolute zero such that magnetic fields cannot exist inside these materials.

Materials with a relative permeability of just greater than one are defined as paramagnetic. In paramagnetic materials, the magnetic moments due to electron orbit and spin are unequal, resulting in a small positive magnetic susceptibility. Magnetization is not significant in paramagnetic materials. Both diamagnetic and paramagnetic materials are typically linear media.

Materials with a relative permeability much greater than one are defined as ferromagnetic. Ferromagnetic materials are always nonlinear. As such, these materials cannot be described by a single value of relative permeability. If a single number is given for the relative permeability of any ferromagnetic material, this number represents an average value of $\mu_r$.

<table>
<thead>
<tr>
<th>Diamagnetic</th>
<th>$\mu_r &lt; 1$</th>
<th>linear</th>
<th>$B = \mu H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paramagnetic</td>
<td>$\mu_r &gt; 1$</td>
<td>linear</td>
<td>$B = \mu (H)H$</td>
</tr>
<tr>
<td>Ferromagnetic</td>
<td>$\mu_r \gg 1$</td>
<td>nonlinear</td>
<td>$B = \mu (H)H$</td>
</tr>
</tbody>
</table>

Ferromagnetic materials lose their ferromagnetic properties at very high temperatures (above a temperature known as the Curie temperature).

The characteristics of ferromagnetic materials are typically presented using the $B$-$H$ curve, a plot of the magnetic flux density $B$ in the material due to a given applied magnetic field $H$. 
The \( B-H \) curve shows the initial magnetization curve along with a curve known as a hysteresis loop. The initial magnetization curve shows the magnetic flux density that would result when an increasing magnetic field is applied to an initially unmagnetized material. An unmagnetized material is defined by the \( B=H=0 \) point on the \( B-H \) curve (no net magnetic flux given no applied field). As the magnetic field increases, at some point, all of the magnetic moments (current loops) within the material align themselves with the applied field and the magnetic flux density saturates \((B_m)\). If the magnetic field is then cycled between the saturation magnetic field value in the forward and reverse directions \((\pm H_m)\), the hysteresis loop results. The response of the material to any applied field depends on the initial state of the material magnetization at that instant.
Magnetic Boundary Conditions

The fundamental boundary conditions involving magnetic fields relate the tangential components of magnetic field and the normal components of magnetic flux density on either side of the media interface. The same techniques used to determine the electric field boundary conditions can be used to determine the magnetic field boundary conditions. The tangential magnetic field boundary condition is found by applying Ampere’s law on a path that straddles the media interface while the normal magnetic flux boundary condition is found by applying Gauss’s law for magnetic fields to a volume straddling the media interface. The resulting boundary conditions are shown below.

Tangential Magnetic Field

\[ \hat{n} \times [H_1 - H_2] = J_s \]

Vector boundary condition relating the magnetic field and surface current at a media interface.

where \( \hat{n} \) is a unit normal to the interface pointing into region 1.
Normal Magnetic Flux Density

The normal components of magnetic flux density are continuous across a media interface.

\[ B_{n1} = B_{n2} \]
**Inductors and Inductance**

An *inductor* is an energy storage device that stores energy in an magnetic field. A inductor typically consists of some configuration of conductor coils (an efficient way of concentrating the magnetic field). Yet, even straight conductors contain inductance. The parameters that define inductors and inductance can be defined as parallel quantities to those of capacitors and capacitance.

<table>
<thead>
<tr>
<th>Inductor</th>
<th>Capacitor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stores energy in a magnetic field</td>
<td>Stores energy in an electric field</td>
</tr>
<tr>
<td>Inductance Definition</td>
<td>Capacitance Definition</td>
</tr>
<tr>
<td>$L \equiv \frac{\Lambda}{I}$</td>
<td>$C \equiv \frac{Q}{V}$</td>
</tr>
<tr>
<td>$L = \text{Inductance (H)}$</td>
<td>$C = \text{Capacitance (F)}$</td>
</tr>
<tr>
<td>$\Lambda = \text{Flux linkage (Wb)}$</td>
<td>$Q = \text{Charge (C)}$</td>
</tr>
<tr>
<td>$I = \text{Current (A)}$</td>
<td>$V = \text{Voltage (V)}$</td>
</tr>
<tr>
<td>$\Lambda = N \int \int B \cdot ds = N \psi_m$</td>
<td>$Q = \int \int D \cdot ds = \psi$</td>
</tr>
<tr>
<td>$W_m = \frac{1}{2} LI^2 = \frac{1}{2} \Lambda I = \frac{1}{2} \frac{\Lambda^2}{L}$</td>
<td>$W_e = \frac{1}{2} CV^2 = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2}{C}$</td>
</tr>
<tr>
<td>$w_m = \frac{1}{2} \mu H^2$</td>
<td>$w_e = \frac{1}{2} \varepsilon E^2$</td>
</tr>
<tr>
<td>$W_m = \int \int \int w_m , dv$</td>
<td>$W_e = \int \int \int w_e , dv$</td>
</tr>
<tr>
<td>$= \frac{1}{2} \int \int \mu H^2 , dv$</td>
<td>$= \frac{1}{2} \int \int \varepsilon E^2 , dv$</td>
</tr>
<tr>
<td>$= \frac{1}{2} \int \int B \cdot H , dv$</td>
<td>$= \frac{1}{2} \int \int D \cdot E , dv$</td>
</tr>
</tbody>
</table>

The *flux linkage* of an inductor defines the total magnetic flux that links the current. If the magnetic flux produced by a given current links that same current, the resulting inductance is defined as a *self inductance*. If the magnetic flux produced by a given current links the current in another circuit, the resulting inductance is defined as a *mutual inductance*. 
Solenoid

A solenoid is a cylindrically shaped current carrying coil. The solenoid is the magnetic field equivalent to the parallel plate capacitor for electric fields. Just as the parallel plate capacitor concentrates the electric field between the plates, the solenoid concentrates the magnetic field within the coil. For the purpose of determining the solenoid magnetic field, the solenoid of length $l$ and radius $a$ which is tightly wound with $N$ turns can be modeled as an equivalent uniform surface current on the cylinder surface.

The equivalent uniform surface current density ($J_o$) for the solenoid is found by spreading the total current of $NI$ over the length $l$.

$$J_s = J_o \frac{\Phi}{l} = \frac{NI}{l} \frac{\Phi}{l}$$

The Biot-Savart law integral to determine the magnetic field of the solenoid equivalent surface current is

$$H = \frac{1}{4\pi} \int \int_S \frac{J_s \times (\mathbf{R} - \mathbf{R}')}{R^3} \, ds'$$
To determine the characteristics of the magnetic field inside the solenoid, we choose the field point $P$ on the solenoid axis ($z$-axis).

\[ J_s = \frac{NI}{l} \hat{\phi} \]

\[ ds' = ad\phi' dz' \]

\[ R = z\hat{z} \]

\[ R' = a\hat{r} + z'\hat{z} \]

\[ R - R' = -a\hat{r} + (z - z')\hat{z} \]

\[ R_o = |R - R'| = \sqrt{a^2 + (z - z')^2} \]

The cross product in the numerator of the Biot-Savart law integral is

\[ J_s \times (R - R') = \left( \frac{NI}{l} \hat{\phi} \right) \times [-a\hat{r} + (z - z')\hat{z}] \]

\[ = \frac{NI}{l} [a\hat{z} + (z - z')\hat{r}] \]

The $\hat{r}$ unit vector written in terms of rectangular coordinate unit vectors is

\[ \hat{r} = \cos \phi' \hat{x} + \sin \phi' \hat{y} \]

The Biot-Savart law integral for the solenoid becomes

\[ H = \frac{1}{4\pi} \int_{-l/2}^{l/2} \int_{-l/2}^{l/2} \frac{NI}{l} [0 (z - z') \cos \phi' \hat{x} + (z - z') \sin \phi' \hat{y} + a\hat{z}] \]

\[ \frac{a^2 + (z - z')^2}{[a^2 + (z - z')^2]^{3/2}} \quad ad\phi' dz' \]

\[ = \frac{NIa^2}{4\pi l} \hat{z} \int_{-l/2}^{l/2} \int_{0}^{2\pi} \frac{d\phi'}{d\phi} \frac{dz'}{[a^2 + (z - z')^2]^{3/2}} \]

\[ = \frac{NIa^2}{4\pi l} \hat{z} (2\pi) \int_{-l/2}^{l/2} \frac{dz'}{[a^2 + (z - z')^2]^{3/2}} \]
The remaining $z'$ integration is the same form as that found for the current segment on the $z$-axis. The result of the integration is

$$H = \frac{NIa^2}{2l} \hat{z} \frac{1}{a^2} \left[ \frac{z + l/2}{\sqrt{a^2 + (z + l/2)^2}} - \frac{z - l/2}{\sqrt{a^2 + (z - l/2)^2}} \right]$$

$$= \frac{NI}{2l} \left[ \frac{z + l/2}{\sqrt{a^2 + (z + l/2)^2}} - \frac{z - l/2}{\sqrt{a^2 + (z - l/2)^2}} \right] \hat{z}$$

The magnetic field at the center of the solenoid ($z = 0$) is

$$H = \frac{NI}{2l} \left[ \frac{l}{\sqrt{a^2 + (l/2)^2}} \right] \hat{z} = \frac{NI}{2 \sqrt{a^2 + (l/2)^2}} \hat{z}$$

At either end of the solenoid ($z = +l/2, -l/2$), the magnetic field is

$$H = \frac{NI}{2l} \left[ \frac{l}{\sqrt{a^2 + l^2}} \right] \hat{z} = \left[ \frac{NI}{2 \sqrt{a^2 + l^2}} \right] \hat{z}$$

For a long solenoid ($l \gg a$), the approximate magnetic field values at the center and at the ends of the solenoid are

$$H \approx \frac{NI}{l} a_z \quad \text{(center)} \quad H \approx \frac{NI}{2l} a_z \quad \text{(at either end)}$$

Thus, the magnetic field at the ends of a long solenoid is approximately half that at the center of the solenoid. However, the magnetic field over the length is a long solenoid is relatively constant. At the ends of the long solenoid, the magnetic field falls rapidly to about one-half of the peak value.
Example (Self inductance / long solenoid)

Given the long solenoid ($l \gg a$), the magnetic field throughout the solenoid can be assumed to be constant.

$$H = \frac{NI}{l} \quad B = \mu H = \mu \frac{NI}{l}$$

According to the definition of inductance, the inductance of the long solenoid is

$$L = \frac{\Lambda}{I} = \frac{N \psi_m}{I}$$

The total magnetic flux through the solenoid is

$$\psi_m = \int \int B \cdot ds = BA = \mu \frac{NI}{l} \pi a^2$$

Inserting the total magnetic flux expression into the inductance equation yields

$$L = \frac{\mu N^2 I \pi a^2}{II} = \frac{\mu N^2 \pi a^2}{l} \quad (H) \quad \text{(long solenoid)}$$

Note that the inductance of the long solenoid is directly proportional to the permeability of the medium inside the core of the solenoid. By using a ferromagnetic material such as iron as the solenoid core, the inductance can be increased significantly given the large relative permeability of a ferromagnetic material.
Toroid

Another commonly encountered magnetic energy storage geometry is the toroid. A toroid is formed by wrapping a conductor around a ring of uniform cross-section (typically circular cross-section). The distance from the center of the ring to the center of the ring cross-section is defined as the mean radius \( r_o \). Given a circular cross-section of radius \( a \), if the mean radius is large relative to the radius of the cross section \( (r_o \gg a) \), then the toroid may be viewed as a long solenoid bent into the shape of a circle (the magnetic field within the toroid may be assumed to be uniform). Application of Ampere’s law on the mean radius path gives

\[
\oint \mathbf{H} \cdot d\mathbf{l} = \oint H_\phi d\mathbf{l} = H_\phi \oint d\mathbf{l} = H_\phi (2\pi r_o) = I_{\text{enclosed}} = NI
\]

Solving for the toroid magnetic field yields

\[
H_\phi = \frac{NI}{2\pi r_o} = \frac{NI}{l}
\]

where \( l = 2\pi r_o \) is the equivalent length of the toroid. The magnetic field at any point within the toroid is the same as that found at the center of the long solenoid. Thus, the self-inductance of the toroid is the same as the equivalent long solenoid (replace \( l \) with \( 2\pi r_o \)).

\[
L = \frac{\mu N^2 \pi a^2}{2\pi r_o} = \frac{\mu N^2 a^2}{2r_o} \quad \text{(toroid)}
\]

The primary advantage of the toroid over the solenoid is the confinement of the magnetic field within the toroid as opposed to the solenoid which produces magnetic fields external to the coil. Also, the toroid does not suffer from the end effects (fringing) seen in the solenoid.
Coaxial Transmission Line

The magnetic field between the conductors of coaxial transmission line was shown using Ampere’s law to be

\[ H = \frac{I}{2\pi r} \hat{\phi} \quad (a < r < b) \]

where \( I \) is the total current in each conductor (flowing in opposite directions in the two conductors). The self inductance of the coaxial transmission line is found by determining the total magnetic flux that links the transmission line current (total flux linkage). The flux linkage for the coaxial transmission line is found by integrating the magnetic flux density between the conductors over the surface \( S \) shown in the figure below.

\[ \Lambda = \psi_m = \int \int_S B \cdot ds \]

\[ B = \frac{\mu I}{2\pi r} \hat{\phi} \]

\[ ds = drdz \hat{\phi} \]

\[ \Lambda = \int \int \frac{\mu I}{2\pi r} \hat{\phi} \cdot drdz \hat{\phi} = \frac{\mu I}{2\pi} \int_a^b \frac{dr}{r} \int_0^l dz = \frac{\mu I l}{2\pi} \ln \left( \frac{b}{a} \right) \]

The inductance of a length \( l \) of coaxial transmission line is

\[ L = \frac{\Lambda}{I} = \frac{\mu I}{2\pi} \ln \left( \frac{b}{a} \right) \quad \text{(H)} \]