Plane Wave Reflection/Transmission at a Dielectric Interface
(Normal Incidence)

When a plane wave propagating in a homogenous medium encounters an interface with a different medium, a portion of the wave is reflected from the interface while the remainder of the wave is transmitted. The reflected and transmitted waves can be determined by enforcing the electromagnetic field boundary conditions at the media interface.

Given a z-directed, x-polarized uniform plane wave incident on a planar media interface located on the x-y plane, the phasor fields associated with the incident, reflected and transmitted fields may be written as

Incident wave fields
\[
\begin{align*}
\vec{E}^i &= E_o e^{-\gamma_1 z} \hat{x} \\
\vec{H}^i &= \frac{E_o}{\eta_1} e^{-\gamma_1 z} \hat{y}
\end{align*}
\]

Reflected wave fields
\[
\begin{align*}
\vec{E}^r &= \Gamma E_o e^{\gamma_1 z} \hat{x} \\
\vec{H}^r &= -\Gamma \frac{E_o}{\eta_1} e^{\gamma_1 z} \hat{y}
\end{align*}
\]

Transmitted wave fields
\[
\begin{align*}
\vec{E}^t &= \tau E_o e^{-\gamma_2 z} \hat{x} \\
\vec{H}^t &= \tau \frac{E_o}{\eta_2} e^{-\gamma_2 z} \hat{y}
\end{align*}
\]

\(\Gamma\) - Reflection coefficient
\(\tau\) - Transmission coefficient
Enforcement of the boundary conditions (continuous tangential electric field and continuous tangential magnetic field) yields

\[ E_x^i + E_x^r = E_x^t \quad \text{at } z = 0 \quad \Rightarrow \quad 1 + \Gamma = \tau \]

\[ H_y^i + H_y^r = H_y^t \quad \text{at } z = 0 \quad \Rightarrow \quad \frac{1 - \Gamma}{\eta_1} = \frac{\tau}{\eta_2} \]

Solving these two equations for the reflection and transmission coefficients gives

\[ \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{(reflection coefficient)} \]

\[ \tau = \frac{2 \eta_2}{\eta_2 + \eta_1} \quad \text{(transmission coefficient)} \]

The total fields in the two media are

\[ \tilde{E}_1 = \tilde{E}^i + \tilde{E}^r = E_o \left[ e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z} \right] \hat{x} \quad \text{(total fields in region 1)} \]

\[ \tilde{H}_1 = \tilde{H}^i + \tilde{H}^r = \frac{E_o}{\eta_1} \left[ e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z} \right] \hat{y} \]

\[ \tilde{E}_2 = \tilde{E}^t = E_o \tau e^{-\gamma_2 z} \hat{x} \quad \text{(total fields in region 2)} \]

\[ \tilde{H}_2 = \tilde{H}^t = \frac{E_o}{\eta_2} \tau e^{-\gamma_2 z} \hat{y} \]

Special cases

\[ \eta_1 = \eta_2 \quad \Gamma = 0 \quad \tau = 1 \quad \text{(total transmission no reflection)} \]

\[ \eta_1 = 0 \quad \Gamma = 1 \quad \tau = 2 \quad \text{(total reflection without inversion of } \tilde{E} \text{)} \]

\[ \eta_2 = 0 \quad \Gamma = -1 \quad \tau = 0 \quad \text{(total reflection with inversion of } \tilde{E} \text{)} \]
Special Case #1 (perfect dielectric/perfect conductor)

region 1  ⇒  lossless dielectric \[ \sigma_1 = 0, \ \eta_1 = \left[ \frac{\mu_2}{\epsilon_1} \right], \ \alpha_1 = 0, \ \gamma = j\beta_1 \]

region 2  ⇒  perfect conductor \[ \sigma_2 = \infty, \ \eta_2 = 0, \ \alpha_2 = \beta_2 \to \infty \]

\[ \eta_2 = 0 \quad \Gamma = -1 \quad \tau = 0 \quad \left( \text{total reflection with inversion of } \vec{E} \right) \]

\[ \vec{E}_1 = E_o \left[ e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z} \right] \hat{x} = E_o \left[ e^{-j\beta_1 z} - e^{j\beta_1 z} \right] \hat{x} = -2jE_o \sin \beta_1 z \hat{x} \]

\[ \vec{H}_1 = \frac{E_o}{\eta_1} \left[ e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z} \right] \hat{y} = \frac{E_o}{\eta_1} \left[ e^{-j\beta_1 z} + e^{j\beta_1 z} \right] \hat{y} = \frac{2E_o}{\eta_1} \cos \beta_1 z \hat{y} \]

\[ \vec{E}_2 = E_o \tau e^{-\gamma_2 z} \hat{x} = 0 \]

\[ \vec{H}_2 = \frac{E_o}{\eta_2} \tau e^{-\gamma_2 z} \hat{y} = 0 \]

If we let \( E_o = |E_o| e^{i\phi} \) and \(-j = e^{-j(\pi/2)}\), the instantaneous fields in the dielectric are

\[ \vec{E}_1 = \text{Re} \left[ \vec{E}_1 e^{j\omega t} \right] = \text{Re} \left[ 2e^{-j\pi/2} |E_o| e^{j\phi} \sin \beta_1 z e^{j\omega t} \hat{x} \right] \]

\[ = 2 |E_o| \sin \beta_1 z \cos(\omega t + \phi - \pi/2) \hat{x} \]

\[ = 2 |E_o| \sin \beta_1 z \sin(\omega t + \phi) \hat{x} \]

\[ \vec{H}_1 = \text{Re} \left[ \vec{H}_1 e^{j\omega t} \right] = \text{Re} \left[ \frac{2|E_o|}{\eta_1} e^{j\phi} \cos \beta_1 z e^{j\omega t} \hat{y} \right] \]

\[ = \frac{2|E_o|}{\eta_1} \cos \beta_1 z \cos(\omega t + \phi) \hat{y} \]

Note that the position dependence of the instantaneous electric and magnetic fields is not a function of time (standing wave).
Assuming for simplicity that $\phi = 0^\circ$ (the phase of the incident electric field is $0^\circ$ at the media interface), the instantaneous electric field in the dielectric is

$$ E_1 = 2|E_o| \sin \beta_1 z \sin \omega t \hat{x} $$

The locations of the minimums (nulls) and maximums (peaks) in the standing wave electric field pattern are found by

$$ |E_1|_{\min} = 0 \text{ occurs when } \sin \beta_1 z = 0 \quad \Rightarrow \quad \beta_1(-z) = n\pi $$

$$ z = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{(2\pi)/\lambda_1} = -n\frac{\lambda_1}{2} \quad n = 0, 1, 2, ... $$

$$ |E_1|_{\max} = 2|E_o| \text{ occurs when } \sin \beta_1 z = 1 \quad \Rightarrow \quad \beta_1(-z) = (2n+1)\frac{\pi}{2} $$

$$ z = -(2n+1)\frac{\pi}{2(\pi/\lambda_1)} = -(2n+1)\frac{\lambda_1}{4} \quad n = 0, 1, 2, ... $$
Special Case #2 (two perfect dielectrics)

region 1 = lossless dielectric \[ \sigma_1 = 0, \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}, \alpha_1 = 0, \gamma = j\beta_1 \]

region 2 = lossless dielectric \[ \sigma_2 = 0, \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}, \alpha_2 = 0, \gamma = j\beta_2 \]

if \( \eta_2 > \eta_1 \)
\[ \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad (0 < \Gamma < 1) \quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} \quad (1 < \tau < 2) \]

if \( \eta_2 < \eta_1 \)
\[ \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad (-1 < \Gamma < 0) \quad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} \quad (0 < \tau < 1) \]

\[ \tilde{E}_1 = E_o \left[ e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z} \right] \hat{x} = E_o \left[ e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z} \right] \hat{x} = E_o e^{-j\beta_1 z} \left[ 1 + \Gamma e^{j2\beta_1 z} \right] \hat{x} \]

\[ \tilde{H}_1 = \frac{E_o}{\eta_1} \left[ e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z} \right] \hat{y} = \frac{E_o}{\eta_1} \left[ e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z} \right] \hat{y} = \frac{E_o}{\eta_1} e^{-j\beta_1 z} \left[ 1 - \Gamma e^{j2\beta_1 z} \right] \hat{y} \]

\[ \tilde{E}_2 = E_o \tau e^{-\gamma_2 z} \hat{x} = E_o \tau e^{-j\beta_2 z} \hat{x} \]

\[ \tilde{H}_2 = \frac{E_o}{\eta_2} \tau e^{-\gamma_2 z} \hat{y} = \frac{E_o}{\eta_2} \tau e^{-j\beta_2 z} \hat{y} \]

Note that standing waves exist only in region 1. The magnitude of the electric field in region 1 can be analyzed to determine the locations of the maximum and minimum values of the electric field standing wave pattern.

\[ |\tilde{E}_1| = |E_o| \left| 1 + \Gamma e^{j2\beta_1 z} \right| \]

\[ \left| 1 + \Gamma e^{j2\beta_1 z} \right| = |1 \angle 0^\circ + \Gamma \angle 2\beta_1 z| \]

The last term above can be described in the complex plane using what is commonly called a crank diagram.
The distance from the origin to the respective point on the circle in the crank diagram represents the magnitude of

\[ 1 + \Gamma e^{j2\beta_1z} \]

If \( \eta_2 > \eta_1 \) (\( \Gamma \) is positive), then the maximum and minimum magnitudes of the function above are

\[ |1 + \Gamma e^{j2\beta_1z}|_{\text{max}} = 1 + |\Gamma| \quad \text{when} \quad 2\beta_1(-z) = n(2\pi) \quad n = 0, 1, 2, ... \]

\[ z = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{(2\pi)/\lambda_1} = -n\frac{\lambda_1}{2} \]

\[ |1 + \Gamma e^{j2\beta_1z}|_{\text{min}} = 1 - |\Gamma| \quad \text{when} \quad 2\beta_1(-z) = (2n + 1)\pi \quad n = 0, 1, 2, ... \]

\[ z = -\frac{(2n + 1)\pi}{2\beta_1} = -\frac{(2n + 1)\pi}{2(2\pi)/\lambda_1} = -(2n + 1)\frac{\lambda_1}{4} \]

\[ |\vec{E}_1|_{\text{max}} = |E_o| \left(1 + |\Gamma|\right) \]

\[ |\vec{E}_1|_{\text{min}} = |E_o| \left(1 - |\Gamma|\right) \]

If \( \eta_1 > \eta_2 \) (\( \Gamma \) is negative), the positions of the maximums and minimums are reversed, but the equations for the maximum and minimum electric field magnitudes in terms of \( |\Gamma| \) are the same.
The **standing wave ratio** \(s\) in a region where standing waves exist is defined as the ratio of the maximum electric field magnitude to the minimum electric field magnitude.

\[
s = \frac{|\vec{E}_1|_{\text{max}}}{|\vec{E}_1|_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \Rightarrow \quad |\Gamma| = \frac{s - 1}{s + 1}
\]

The standing wave ratio (purely real) ranges from a minimum value of 1 (no reflection, \(|\Gamma| = 0\)) to \(\infty\) (total reflection, \(|\Gamma| = 1\)). The standing wave ratio is sometimes defined as dB as

\[
s(\text{dB}) = 20 \log_{10} s
\]

**Example** (Plane wave reflection/transmission)

A uniform plane wave in air is normally incident on an infinite lossless dielectric material having \(\varepsilon = 3\varepsilon_o\) and \(\mu = \mu_o\). If the incident wave is \(E^i = 10 \cos(\omega t - z)\hat{y}\) V/m, find (a.) \(\omega\) and \(\lambda\) of the waves in both regions, (b.) \(H^i\) (c.) \(\Gamma\) and \(\tau\) and (d.) The total electric field and time-average power in both regions.

Region 1 \([z < 0]\) (Air) \((\mu_1 = \mu_o, \varepsilon_1 = \varepsilon_o, \sigma_1 = 0)\)

\[
\begin{align*}
\alpha_1 &= 0, \quad \beta_1 = \omega \sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} \\
\gamma_1 &= j\beta_1, \quad \eta_1 = \sqrt{\frac{\mu_o}{\varepsilon_o}} = \eta_o \\
u_1 &= c
\end{align*}
\]

Medium 2 \([z > 0]\) (Dielectric) \((\mu_2 = \mu_o, \varepsilon_2 = 3\varepsilon_o, \sigma_2 = 0)\)

\[
\begin{align*}
\alpha_2 &= 0, \quad \beta_2 = \omega \sqrt{3\mu_o \varepsilon_o} = \sqrt{3} \frac{\omega}{c} \\
\gamma_2 &= j\beta_2, \quad \eta_2 = \sqrt{\frac{\mu_o}{3\varepsilon_o}} = \frac{\eta_o}{\sqrt{3}} \\
u_2 &= \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{\sqrt{3}}
\end{align*}
\]
(a.) \[ E^i = 10 \cos(\omega t - z) \hat{y} \quad \Rightarrow \quad \tilde{E}^i = 10 e^{-jz} \hat{y} = E_o e^{-j\beta_1 z} \hat{y} \]
\[ E_o = 10 \quad \beta_1 = 1 \]
\[ \beta_1 = \frac{2\pi}{\beta_1} = \frac{\omega}{u_1} = \frac{\omega}{c} = 1 \text{ rad/m} \quad \beta_2 = \frac{2\pi}{\beta_2} = \frac{\omega}{u_2} = \frac{\omega}{\sqrt{3}c} = \sqrt{3} \beta_1 \text{ rad/m} \]
\[ \lambda_1 = \frac{2\pi}{\beta_1} = 2\pi = 6.28 \text{ m} \quad \lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = 3.63 \text{ m} \]
\[ \omega = \beta_1 u_1 = \beta_2 u_2 = 3 \times 10^8 \text{ rad/s} (47.8 \text{ MHz}) \]

(b.) \[ \tilde{H}^i = \frac{E_o}{\eta_o} e^{-j\beta_1 z} (-\hat{x}) = -\frac{10}{377} e^{-jz} \hat{x} = -0.0266 e^{-jz} \hat{x} \]
\[ H^i = \text{Re} \left\{ -0.0266 e^{-jz} e^{j\omega t} \hat{x} \right\} = -0.0266 \cos(\omega t - z) \hat{x} \quad (\text{A/m}) \]

(c.) \[ \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_o - \eta_o}{\sqrt{3} + \eta_o} = -0.268 \]
\[ \tau = 1 + \Gamma = 0.732 \]

(d.) \[ \tilde{E}_1 = E_o \left[ e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z} \right] \hat{y} \quad \tilde{E}_2 = \tau E_o e^{-j\beta_2 z} \hat{y} \]
\[ = \left[ 10 e^{-jz} - 2.68 e^{jz} \right] \hat{y} \quad \hat{y} = 7.32 e^{-j\sqrt{3}z} \hat{y} \]
\[ E_1 = \text{Re} \left\{ (10 e^{-jz} - 2.68 e^{jz}) e^{j\omega t} \hat{y} \right\} \]
\[ = \left[ 10 \cos(\omega t - z) - 2.68 \cos(\omega t + z) \right] \hat{y} \quad \text{V/m} \]
\[ E_2 = \text{Re} \left\{ 7.32 e^{-j\sqrt{3}z} e^{j\omega t} a_y \right\} \]
\[ = \left[ 7.32 \cos(\omega t - \sqrt{3}z) \right] \hat{y} \quad \text{V/m} \]
The time average power density in region 1 is due to the +z-directed incident wave and the −z-directed reflected wave. The time-average power density in region 2 is due to the +z-directed transmitted wave.

\[ P_{\text{ave},1} = \frac{|\vec{E}^i|^2}{2\eta_1} \hat{z} + \frac{|\vec{E}^r|^2}{2\eta_1} (-\hat{z}) \]

\[ = \frac{(10)^2 - (2.68)^2}{2(377)} \hat{z} \]

\[ = \frac{123}{377} \frac{\text{mW}}{\text{m}^2} \]

\[ P_{\text{ave},2} = \frac{|\vec{E}^t|^2}{2\eta_2} \hat{z} = \frac{(7.32)^2}{2(377/\sqrt{3})} \hat{z} \]

\[ = \frac{123}{377/\sqrt{3}} \frac{\text{mW}}{\text{m}^2} \]