CROSSTALK

Given a device with wires or PCB lands in close proximity carrying distinct signals, the signals propagating on their designated conductors can be unintentionally coupled to nearby conductors. This process is known as crosstalk. Crosstalk is intra-system interference (the emitter and the receptor are located in the same device) that can negatively affect the operation of the product. Unlike the far-fields encountered in the radiated emissions, crosstalk involves near-field electromagnetic coupling.

Crosstalk commonly occurs in cable bundles or multiconductor transmission lines. The multiconductor transmission line typically uses multiple conductors along with a common reference conductor to transmit distinct signals along the line. A minimum of three conductors is necessary in order for crosstalk to occur. Some examples of three-conductor transmission lines are shown below.
CROSSTALK ON THREE-CONDUCTOR TRANSMISSION LINES

Consider the three-conductor transmission line model shown below. The three-conductor line consists of an emitter conductor, a receptor conductor and a reference conductor.

The circuit parameters for the three-conductor line will be written in the time-domain according to standard transmission line theory. The emitter conductor and the reference conductor form the normal two-conductor transmission line connection between the voltage source $v_s(t)$ and the load $R_L$. The voltage and current along the emitter conductor, which are functions of time and position, are designated as $v_E(z,t)$ and $i_E(z,t)$, respectively. The signal traveling on the emitter/reference conductor pair will generate a (near) field that will couple onto the two-conductor transmission line formed by the receptor/reference conductor pair. The intended signal traveling on the receptor/reference conductor pair is ignored for this analysis since we are only concerned with the signal induced on this conductor pair. Nor are we concerned with the location of the source and load connections on the receptor/reference conductor pair (the source could be located at either end). For this reason, we designate the end of the receptor/reference conductor pair closest to the voltage source $v_s(t)$ as the near end while the opposite end is designated as the far end.
The near end and the far end of the receptor/reference conductor pair are terminated with resistances $R_{NE}$ and $R_{FE}$, respectively. These resistances represent the input impedances of whatever circuitry may be connected to the ends of this conductor pair. Note that all of the terminations in our three-conductor transmission line model have been assumed to be resistive. This is only a simplifying assumption and the results we obtain for this transmission line can be applied to complex terminations.

If the signals on the three-conductor transmission line are assumed to be transverse electromagnetic (TEM) modes, the current and voltage at any point on the emitter and receptor transmission lines can be uniquely defined. The transverse field distributions associated with the TEM modes are identical to the fields produced on the same three-conductor line geometry under static conditions. Although the actual modes encountered on a typical three-conductor transmission line are not strictly TEM (transmission lines with inhomogeneous dielectrics cannot support true TEM waves), the transverse fields associated with the quasi-TEM modes can be accurately approximated by pure TEM modes. Thus, the quasi-TEM approximation is assumed when performing a crosstalk analysis on the three-conductor transmission line.

The equations defining the TEM mode voltage and current at any point on a uniform two-conductor transmission line (commonly called the transmission line equations) are derived by modeling a short segment of the transmission line by an equivalent circuit. The equivalent circuit of the two-conductor transmission line includes four per-unit length parameters ($r$, $l$, $c$, and $g$) as defined below.

\[
\begin{align*}
    r & = \text{series resistance per unit length (}\Omega/\text{m}) \text{ of the transmission line conductors.} \\
    l & = \text{series inductance per unit length (H/m) of the transmission line conductors (internal plus external inductance).} \\
    g & = \text{shunt conductance per unit length (S/m) of the media between the transmission line conductors (insulator leakage current).} \\
    c & = \text{shunt capacitance per unit length (F/m) between the transmission line conductors.}
\end{align*}
\]
Note that each of the per-unit-length parameters must be multiplied by the length of the transmission line segment to yield the equivalent circuit component values in ohms, henries, farads and siemens. The transmission line equations relating the current and voltage at the input and output of the equivalent circuit are obtained by simple circuit analysis.

The concept of the equivalent circuit for the two-conductor transmission line can be extended to the three-conductor line with a few minor modifications. The three-conductor transmission line is simply modeled as two separate two-conductor transmission lines (emitter line and receptor line) utilizing a common reference conductor. Thus, the emitter and receptor conductors each include a series resistance \((r_E \Delta z, r_R \Delta z)\) and inductance \((l_E \Delta z, l_R \Delta z)\). In the two wire transmission line, the resistance of both conductors is included in the \(r \Delta z\) term. Since the three-conductor line shares a common reference conductor carrying the return current of both the emitter and receptor lines, the resistance of the reference conductor is included separately \((r_o \Delta z)\). The inductive coupling of the emitter and receptor lines is included in a mutual coupling term \((l_m \Delta z)\) between the series inductances of the two lines.
The parallel shunt impedance of the two-wire line \((c \Delta z\) in parallel with \(g \Delta z\)) must be included for each conductor pair in the three-wire line. That is, there is capacitance and conductance between the emitter-reference conductor pair \((c_E \Delta z, g_E \Delta z)\), the receptor-reference conductor pair \((c_R \Delta z, g_R \Delta z)\), and mutual capacitance and conductance between the emitter-receptor conductor pair \((c_m \Delta z, g_m \Delta z)\).

![Equivalent circuit of a three-conductor transmission line segment of length \(\Delta z\).](image)

The development of the transmission line equations for the three-conductor transmission line equivalent circuit involves application of Kirchoff’s voltage law (KVL) to the emitter and receptor input voltages relative to the reference conductor \([v_E(z,t) and v_R(z,t)]\) and Kirchoff’s current law (KCL) to the emitter and receptor input currents \([i_E(z,t) and i_R(z,t)]\).
Application of KVL to \( v_E(z,t) \) and \( v_R(z,t) \) yields

\[
v_E(z,t) = r_E \Delta z i_E(z,t) + l_E \Delta z \frac{\partial i_E(z,t)}{\partial t} + l_m \Delta z \frac{\partial i_R(z,t)}{\partial t} + v_E(z+\Delta z,t) + r_o \Delta z [i_E(z,t) + i_R(z,t)]
\]

\[
v_R(z,t) = r_R \Delta z i_R(z,t) + l_R \Delta z \frac{\partial i_R(z,t)}{\partial t} + l_m \Delta z \frac{\partial i_E(z,t)}{\partial t} + v_R(z+\Delta z,t) + r_o \Delta z [i_E(z,t) + i_R(z,t)]
\]

Grouping the terms involving \( v_E(z,t) \) and \( v_R(z,t) \) on the same side of the equations, dividing both sides of the equations by \( \Delta z \), and taking the limits as \( \Delta z \) approaches zero gives

\[
\lim_{\Delta z \to 0} \left[ \frac{v_E(z+\Delta z,t) - v_E(z,t)}{\Delta z} \right] = -(r_E + r_o) i_E(z,t) - r_o i_R(z,t)
\]

\[
- l_E \frac{\partial i_E(z,t)}{\partial t} - l_m \frac{\partial i_R(z,t)}{\partial t}
\]

\[
\lim_{\Delta z \to 0} \left[ \frac{v_R(z+\Delta z,t) - v_R(z,t)}{\Delta z} \right] = -r_o i_E(z,t) - (r_E + r_o) i_R(z,t)
\]

\[
- l_m \frac{\partial i_E(z,t)}{\partial t} - l_R \frac{\partial i_R(z,t)}{\partial t}
\]

The limits on the left-hand side of the equations above can be identified as the partial derivatives of \( v_E(z,t) \) and \( v_R(z,t) \) with respect to \( z \). Inserting the derivatives with respect to position gives two of the four transmission line equations for the three-conductor line.
Application of KCL to \( i_E(z,t) \) and \( i_R(z,t) \) yields

\[
\frac{\partial v_E(z,t)}{\partial z} = - (r_E + r_o) i_E(z,t) - r_o i_R(z,t) - I_E \frac{\partial i_E(z,t)}{\partial t} - I_m \frac{\partial i_R(z,t)}{\partial t}
\]

\[
\frac{\partial v_R(z,t)}{\partial z} = - r_o i_E(z,t) - (r_E + r_o) i_R(z,t) - I_m \frac{\partial i_E(z,t)}{\partial t} - I_R \frac{\partial i_R(z,t)}{\partial t}
\]

Grouping the terms involving \( i_E(z,t) \) and \( i_R(z,t) \) on the same side of the equations, dividing both sides of the equations by \( \Delta z \), and taking the limits as \( \Delta z \) approaches zero gives

\[
i_E(z,t) - g_m \Delta z [v_E(z + \Delta z,t) - v_R(z + \Delta z,t)]
\]

\[
- c_m \Delta z \frac{\partial [v_E(z + \Delta z,t) - v_R(z + \Delta z,t)]}{\partial t}
\]

\[
- g_E \Delta z v_E(z + \Delta z,t) - c_E \Delta z \frac{\partial v_E(z + \Delta z,t)}{\partial t}
\]

\[
= i_E(z + \Delta z,t)
\]

\[
i_R(z,t) - g_R \Delta z v_R(z + \Delta z,t) - c_R \Delta z \frac{\partial v_R(z + \Delta z,t)}{\partial t}
\]

\[
- c_m \Delta z \frac{\partial [v_R(z + \Delta z,t) - v_E(z + \Delta z,t)]}{\partial t}
\]

\[
- g_m \Delta z [v_R(z + \Delta z,t) - v_E(z + \Delta z,t)]
\]

\[
= i_R(z + \Delta z,t)
\]
The limits on the left-hand side of the equations above can be identified as the partial derivatives of $i_E(z,t)$ and $i_R(z,t)$ with respect to $z$. Inserting the derivatives with respect to position gives the remaining two transmission line equations for the three-conductor line.

\[
\lim_{\Delta z \to 0} \left[ \frac{i_E(z + \Delta z, t) - i_E(z, t)}{\Delta z} \right] = -\left( g_E + g_m \right) v_E(z, t) + g_m v_R(z, t) \\
\quad - \left( c_E + c_m \right) \frac{\partial v_E(z, t)}{\partial t} + c_m \frac{\partial v_R(z, t)}{\partial t}
\]

\[
\lim_{\Delta z \to 0} \left[ \frac{i_R(z + \Delta z, t) - i_R(z, t)}{\Delta z} \right] = -g_m v_E(z, t) - \left( g_R + g_m \right) v_R(z, t) \\
\quad + c_m \frac{\partial v_E(z, t)}{\partial t} - \left( c_R + c_m \right) \frac{\partial v_R(z, t)}{\partial t}
\]

The limits on the left-hand side of the equations above can be identified as the partial derivatives of $i_E(z,t)$ and $i_R(z,t)$ with respect to $z$. Inserting the derivatives with respect to position gives the remaining two transmission line equations for the three-conductor line.

\[
\frac{\partial i_E(z,t)}{\partial z} = -(g_E + g_m) v_E(z, t) + g_m v_R(z, t) - (c_E + c_m) \frac{\partial v_E(z, t)}{\partial t} + c_m \frac{\partial v_R(z, t)}{\partial t}
\]

\[
\frac{\partial i_R(z,t)}{\partial z} = g_m v_E(z, t) - (g_R + g_m) v_R(z, t) + c_m \frac{\partial v_E(z, t)}{\partial t} - (c_R + c_m) \frac{\partial v_R(z, t)}{\partial t}
\]

The four equations which make up the transmission line equations for the three-conductor transmission line can be written concisely using matrix notation. The four unknowns are the voltages $v_E(z,t)$ and $v_R(z,t)$ and the currents $i_E(z,t)$ and $i_R(z,t)$. The four equations can be written as a pair of matrix equations when 2-element voltage and current vectors are defined according to

\[
v(z,t) = \begin{bmatrix} v_E(z,t) \\ v_R(z,t) \end{bmatrix} \quad i(z,t) = \begin{bmatrix} i_E(z,t) \\ i_R(z,t) \end{bmatrix}
\]
The resulting three-conductor transmission line equations are

\[
\frac{\partial v(z,t)}{\partial z} = -r i(z,t) - l \frac{\partial i(z,t)}{\partial t} \\
\frac{\partial i(z,t)}{\partial z} = -g v(z,t) - c \frac{\partial v(z,t)}{\partial t}
\]

where the per-unit-length matrices are given by

\[
p = \begin{bmatrix}
r_E + r_o & r_o \\
r_o & r_R + r_o
\end{bmatrix}
\]

\[
l = \begin{bmatrix}
l_E & l_m \\
l_m & l_R
\end{bmatrix}
\]

\[
g = \begin{bmatrix}
g_E + g_m & -g_m \\
-g_m & g_R + g_m
\end{bmatrix}
\]

\[
c = \begin{bmatrix}
c_E + c_m & -c_m \\
-c_m & c_R + c_m
\end{bmatrix}
\]

The general form of the three-conductor line transmission line equations are identical to those of the two-conductor line (the 2×2 per-unit-length matrices of the three-conductor line reduce to single elements for the two-conductor line). This property extends to multi-conductor transmission lines with \(N\) conductors (using a common reference conductor) where the dimension of the per-unit-length matrices is \((N-1)\times(N-1)\).

The three-conductor line equations given above represent the general time-domain equations that characterize crosstalk on this transmission line. The corresponding frequency-domain equations are obtained by transforming the time-domain equations into the frequency domain based on the relationship between the instantaneous and phasor quantities.
The instantaneous (time-domain) voltage and current vectors are related to the phasor (frequency-domain) voltage and current vectors according to the following.

\[
\mathbf{v}(z,t) = \text{Re}\left\{ \mathbf{\hat{V}}(z) e^{j\omega t} \right\}
\]

\[
\mathbf{i}(z,t) = \text{Re}\left\{ \mathbf{\hat{I}}(z) e^{j\omega t} \right\}
\]

where the phasor voltage and current vectors are defined by

\[
\mathbf{\hat{V}}(z) = \begin{bmatrix} \mathbf{\hat{V}}_E(z) \\ \mathbf{\hat{V}}_R(z) \end{bmatrix} \quad \mathbf{\hat{I}}(z) = \begin{bmatrix} \mathbf{\hat{I}}_E(z) \\ \mathbf{\hat{I}}_R(z) \end{bmatrix}
\]

The phasor voltage and current that correspond to the time-derivatives of the voltage and current in the three-conductor line time-domain equations are found by differentiating the time-domain/frequency domain transformation with respect to time.

\[
\frac{\partial \mathbf{v}(z,t)}{\partial t} = \text{Re}\left\{ j\omega \mathbf{\hat{V}}(z) e^{j\omega t} \right\}
\]

\[
\frac{\partial \mathbf{i}(z,t)}{\partial t} = \text{Re}\left\{ j\omega \mathbf{\hat{I}}(z) e^{j\omega t} \right\}
\]

\[
\frac{\partial \mathbf{v}(z,t)}{\partial t} \Leftrightarrow j\omega \mathbf{\hat{V}}(z)
\]

\[
\frac{\partial \mathbf{i}(z,t)}{\partial t} \Leftrightarrow j\omega \mathbf{\hat{I}}(z)
\]

Thus, the phasors corresponding to the time derivatives of the voltage and the current are simply the voltage and current phasors multiplied by \((j\omega)\). The spatial derivatives of the voltage and current in the frequency-domain can be written as ordinary derivatives since the voltage and current phasors are functions of \(z\) only.
The phasor form of the three-conductor transmission line matrix equations are

\[
\frac{d\hat{V}(z)}{dz} = -(r + j\omega l)\hat{f}(z) = -\hat{Z}\hat{f}(z)
\]

\[
\frac{d\hat{f}(z)}{dz} = -(g + j\omega c)\hat{V}(z) = -\hat{Y}\hat{V}(z)
\]

Note that these equations are coupled first-order differential equations. The impedance and admittance matrices can be written as

\[
\hat{Z} = \begin{bmatrix}
r_E + r_o + j\omega l_E & r_o + j\omega l_m \\
r_o + j\omega l_m & r_R + r_o + j\omega l_R
\end{bmatrix}
\]

\[
\hat{Y} = \begin{bmatrix}
g_E + g_m + j\omega (c_E + c_m) & -(g_m + j\omega c_m) \\
-(g_m + j\omega c_m) & g_R + g_m + j\omega (c_R + c_m)
\end{bmatrix}
\]

The equations can be written in terms of voltage or current alone by differentiating both sides of the equations and inserting the opposite equation prior to differentiation.

\[
\frac{d^2\hat{V}(z)}{dz^2} = -\hat{Z} \frac{d\hat{f}(z)}{dz} = -\hat{Y} \hat{V}(z)
\]

\[
\frac{d^2\hat{f}(z)}{dz^2} = -\hat{Y} \frac{d\hat{V}(z)}{dz} = -\hat{Z} \hat{f}(z)
\]

The first matrix equation above defines a pair of equations involving the voltages along the emitter and receptor lines of the three-conductor line. The second matrix equation defines a pair of equations involving the currents along the emitter and receptor lines of the three-conductor line. Thus, we must solve a $2 \times 2$ linear system for the coupled voltage pair or the coupled current pair.
We would like to incorporate the general solutions for the voltage and current along the individual emitter and receptor lines (two-conductor solutions for the voltage and current on an isolated emitter line and an isolated receptor line) into our three-conductor line solution. In this way, the differential equations for the three-conductor line voltage and current can be totally decoupled. Using the current equation, the three-conductor line currents can be defined in terms of two-conductor modal currents on the emitter and receptor lines according to a matrix transformation which satisfies

\[ \hat{\mathbf{f}}(z) = \hat{T} \hat{\mathbf{f}}_m(z) \]

or

\[
\begin{bmatrix}
\hat{I}_E(z) \\
\hat{I}_R(z)
\end{bmatrix} = \hat{T} \begin{bmatrix}
\hat{I}_{mE}(z) \\
\hat{I}_{mR}(z)
\end{bmatrix}
\]

where the transformation matrix is 2×2. Inserting this solution into the current linear system gives

\[
\hat{T} \frac{d^2 \hat{I}_m(z)}{dz^2} = \hat{\mathbf{Y}} \hat{\mathbf{Z}} \hat{T} \hat{\mathbf{f}}_m(z)
\]

\[
\frac{d^2 \hat{I}_m(z)}{dz^2} = \hat{T}^{-1} \hat{\mathbf{Y}} \hat{\mathbf{Z}} \hat{T} \hat{\mathbf{f}}_m(z)
\]

The modal currents can be totally uncoupled if a transformation can be found that diagonalizes the matrix on the right hand side of the equation above. The diagonalization is defined by

\[
\hat{T}^{-1} \hat{\mathbf{Y}} \hat{\mathbf{Z}} \hat{T} = \hat{\mathbf{\bar{\gamma}}^2} = \begin{bmatrix}
\hat{\gamma}_E^2 & 0 \\
0 & \hat{\gamma}_R^2
\end{bmatrix}
\]
The diagonalization yields two decoupled second order differential equations for the modal currents.

\[ \frac{d^2 \hat{I}_{mE}(z)}{dz^2} = \gamma^2_E \hat{I}_{mE}(z) \]

\[ \frac{d^2 \hat{I}_{mR}(z)}{dz^2} = \gamma^2_R \hat{I}_{mR}(z) \]

The constants on the right hand side of the equations above are the eigenvalues of the admittance matrix/impedance matrix product and represent the propagation constants for the individual modal currents. The differential equations above are identical to those describing the solution current (and voltage) along a two-conductor line. The general solutions for the modal currents on the emitter and receptor lines can thus be written in terms of currents associated with forward and reverse waves:

\[ \hat{I}_{mE}(z) = \hat{I}_{mE}^+ e^{-\gamma_E z} - \hat{I}_{mE}^- e^{\gamma_E z} \]

\[ \hat{I}_{mR}(z) = \hat{I}_{mR}^+ e^{-\gamma_R z} - \hat{I}_{mR}^- e^{\gamma_R z} \]

where the current coefficients of the modal currents are determined according to the line connections. The modal currents can be written in matrix form as

\[ \hat{I}_m(z) = e^{-\gamma z} \hat{I}_m^+ - e^{\gamma z} \hat{I}_m^- \]

where

\[ e^{\pm \gamma z} = \begin{bmatrix} e^{\pm \gamma_E z} & 0 \\ 0 & e^{\pm \gamma_R z} \end{bmatrix} \]

\[ \hat{I}^\pm_m = \begin{bmatrix} \hat{I}_{mE}^\pm \\ \hat{I}_{mR}^\pm \end{bmatrix} \]

The three-conductor line currents in terms of the modal currents are

\[ \hat{I}(z) = \hat{F} \hat{I}_m(z) = \hat{F} (e^{-\gamma z} \hat{I}_m^+ - e^{\gamma z} \hat{I}_m^-) \]
The three-conductor line voltages are found according to
\[
\frac{d\hat{V}(z)}{dz} = -\hat{\gamma} \hat{V}(z)
\]
\[
\hat{V}(z) = -\hat{\gamma}^{-1} \frac{d\hat{I}(z)}{dz} = \hat{\gamma}^{-1} \hat{\gamma} (e^{-\gamma z} \hat{I}_m^+ + e^{\gamma z} \hat{I}_m^-)
\]

A comparison of the three-conductor line voltage equation to that of a two-conductor line suggests that a characteristic impedance matrix can be identified if the voltage equation can be expressed as
\[
\hat{V}(z) = \hat{\mathcal{Z}}_o \hat{\gamma} (e^{-\gamma z} \hat{I}_m^+ + e^{\gamma z} \hat{I}_m^-)
\]
The definition of the propagation constant matrix can be manipulated into the proper form.
\[
\hat{\gamma}^{-1} \hat{\gamma} \hat{\gamma}^{-1} \hat{\gamma} = \hat{\gamma}^2
\]
\[
\hat{\mathcal{Z}} \hat{\gamma} \hat{\gamma}^{-1} = \hat{\gamma}^{-1} \hat{\gamma}
\]
Inserting this result into the voltage equation gives
\[
\hat{V}(z) = (\hat{\mathcal{Z}} \hat{\gamma} \hat{\gamma}^{-1} \hat{\gamma}) \hat{\gamma} (e^{-\gamma z} \hat{I}_m^+ + e^{\gamma z} \hat{I}_m^-)
\]
so that characteristic impedance matrix can be identified as
\[
\hat{\mathcal{Z}}_o = \hat{\mathcal{Z}} \hat{\gamma} \hat{\gamma}^{-1} \hat{\gamma}^{-1}
\]
Thus, the voltage and current along the three-conductor line can be written in terms of two equations defined by
\[
\hat{V}(z) = \hat{\mathcal{Z}}_o \hat{\gamma} (e^{-\gamma z} \hat{I}_m^+ + e^{\gamma z} \hat{I}_m^-)
\]
\[
\hat{I}(z) = \hat{\gamma} (e^{-\gamma z} \hat{I}_m^+ - e^{\gamma z} \hat{I}_m^-)
\]
These equations are identical in form to those of the two-conductor line. In order to apply these equations, the four current coefficients contained in the four equations of the linear system must be determined.
The unknown coefficients can be determined according to the connection of the three-conductor transmission line. The assumed connection of the three-conductor line for crosstalk calculations is shown below. The phasor quantities which appear in the three-conductor line equations are labeled on the figure.

The voltages at the four ports of the three-conductor line are defined by

\[
\begin{align*}
\hat{V}_E(0) &= \hat{V}_S - \hat{I}_E(0) R_S & \hat{V}_R(\infty) = \hat{I}_E(\infty) R_L \\
\hat{V}_R(0) &= -\hat{I}_R(0) R_{NE} & \hat{V}_R(\infty) = \hat{I}_R(\infty) R_{FE}
\end{align*}
\]

which can be written in matrix form as

\[
\begin{align*}
\hat{V}(0) &= \hat{V}_S - \hat{Z}_S \hat{I}(0) \\
\hat{V}(\infty) &= \hat{Z}_L \hat{I}(\infty)
\end{align*}
\]

where

\[
\begin{align*}
\hat{V}_S &= \begin{bmatrix} \hat{V}_S \\ 0 \end{bmatrix} & \hat{Z}_S &= \begin{bmatrix} R_S & 0 \\ 0 & R_{NE} \end{bmatrix} & \hat{Z}_L &= \begin{bmatrix} R_L & 0 \\ 0 & R_{FE} \end{bmatrix}
\end{align*}
\]
Evaluating the three-conductor transmission line equations at the port locations \((z = 0\) and \(z = L\)) gives

\[
\hat{V}(0) = \hat{Z}_o \hat{F} (\hat{I}_m^* + \hat{I}_m)
\]

\[
\hat{I}(0) = \hat{F} (\hat{I}_m^* - \hat{I}_m)
\]

\[
\hat{V}(L) = \hat{Z}_o \hat{F} (e^{-\gamma_L L} \hat{I}_m^* + e^{\gamma_L L} \hat{I}_m)
\]

\[
\hat{I}(L) = \hat{F} (e^{-\gamma_L L} \hat{I}_m^* - e^{\gamma_L L} \hat{I}_m)
\]

Inserting these equations into the port voltage expressions yields

\[
\hat{Z}_o \hat{F} (\hat{I}_m^* + \hat{I}_m) = \hat{V}_S - \hat{Z}_S \hat{F} (\hat{I}_m^* - \hat{I}_m)
\]

\[
\hat{Z}_o \hat{F} (e^{-\gamma_L L} \hat{I}_m^* + e^{\gamma_L L} \hat{I}_m) = \hat{Z}_L \hat{F} (e^{-\gamma_L L} \hat{I}_m^* - e^{\gamma_L L} \hat{I}_m)
\]

These two matrix equations can be written as

\[
\begin{bmatrix}
\hat{X}_o \hat{F} + \hat{Z}_S \hat{F} & \hat{X}_o \hat{F} - \hat{Z}_S \hat{F} \\
\hat{X}_o \hat{F} - \hat{Z}_L \hat{F} e^{\gamma_L L} & \hat{X}_o \hat{F} - \hat{Z}_L \hat{F} e^{\gamma_L L}
\end{bmatrix}
\begin{bmatrix}
\hat{I}_m^*(z) \\
\hat{I}_m(z)
\end{bmatrix}
= 
\begin{bmatrix}
\hat{V}_S \\
0
\end{bmatrix}
\]

We may then solve the linear system of four equations for the four unknown current coefficients. Given these coefficients, the current and voltage at any location along the emitter or receptor lines can be determined.

For crosstalk problems, we typically are only interested in the voltages and currents at the ends of the transmission line (four ports). Thus, the same equations used to derive the current coefficient linear system can be used to define a linear system in terms of the port voltages and currents (by eliminating the current coefficients). The result is a matrix that relates the voltages and currents at the far end \((z = L)\) to the voltages and the currents at the near end \((z = 0)\).
The equations relating the voltages and currents at the four ports can be inserted into the equations above to yield a linear system in terms of the four port currents.

\[
\begin{bmatrix}
\hat{V}(\mathcal{L}) \\
\hat{I}(\mathcal{L})
\end{bmatrix} =
\begin{bmatrix}
\hat{\phi}_{11} & \hat{\phi}_{12} \\
\hat{\phi}_{21} & \hat{\phi}_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{V}(0) \\
\hat{I}(0)
\end{bmatrix}
\]

\[
\hat{\phi}_{11} = \frac{1}{2} \hat{Z}^{-1} \hat{f}(e^{\gamma z} + e^{-\gamma z}) \hat{Z}^{-1} \hat{f}
\]

\[
\hat{\phi}_{12} = -\frac{1}{2} \hat{Z}^{-1} \hat{f}(e^{\gamma z} - e^{-\gamma z}) \hat{Z}^{-1}
\]

\[
\hat{\phi}_{21} = -\frac{1}{2} \hat{Z}^{-1} \hat{f}(e^{\gamma z} - e^{-\gamma z}) \hat{Z}^{-1} \hat{f}
\]

\[
\hat{\phi}_{22} = \frac{1}{2} \hat{Z}^{-1} \hat{f}(e^{\gamma z} + e^{-\gamma z}) \hat{Z}^{-1}
\]

The equations relating the voltages and currents at the four ports can be inserted into the equations above to yield a linear system in terms of the four port currents.

\[
\begin{bmatrix}
\hat{Z_L} \hat{I}(\mathcal{L}) \\
\hat{I}(\mathcal{L})
\end{bmatrix} =
\begin{bmatrix}
\hat{\phi}_{11} & \hat{\phi}_{12} \\
\hat{\phi}_{21} & \hat{\phi}_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{V}_S - \hat{Z}_S \hat{I}(0) \\
\hat{I}(0)
\end{bmatrix}
\]

This linear system can be solved first for the port currents at \( z = 0 \) from which the solutions for the port currents at \( z = \mathcal{L} \) can be obtained.
MULTICONDUCTOR TRANSMISSION LINE
PER-UNIT-LENGTH PARAMETERS

The per-unit-length parameter matrices $r$, $l$, $g$, and $c$ are required in the determination of crosstalk responses in multiconductor transmission lines. The multiconductor transmission line parameters that depend on the internal distribution of the conductor currents over the conductor cross-sections (resistance and internal inductance) can be determined using the same equations used for the two conductor line. The conductors of the multiconductor transmission line are assumed to be sufficiently spaced such that the proximity effect is insignificant. The internal inductance of the multiconductor transmission line conductors can be neglected since it is typically a small fraction of the corresponding external inductance.

As previously shown, the per-unit-length parameters $r$, $l$, $g$, and $c$ for a two-conductor line in a homogeneous medium characterized by $(\mu, \epsilon, \sigma)$ are related by

\[ lc = \mu \epsilon \]
\[ lg = \mu \sigma \]

where the per-unit-length internal inductance is assumed to be negligible. According to these relationships, only the per-unit-length inductance is required to obtain all three external parameters since the per-unit-length capacitance and conductance can both be determined from the per-unit-length inductance. The per-unit-length parameter matrices $r$, $l$, $g$, and $c$ for a multiconductor transmission line located in a homogeneous medium satisfy the same relationships in matrix form.

\[ lc = cl = \mu \epsilon I \]
\[ lg = gl = \mu \sigma I \]

where $I$ is the identity matrix defined by

\[
I = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]
Thus, the per-unit-length capacitance and conductance matrices can be determined from the per-unit-length inductance matrix according to

\[ c = \mu \varepsilon I^{-1} \]
\[ g = \mu \sigma I^{-1} \]

The determination of per-unit-length matrices for multiconductor transmission lines with inhomogeneous dielectrics normally requires a numerical technique.

Consider the three conductor line shown below consisting of three round wires. The emitter, receptor and reference conductors have radii of \( a_E, a_R \) and \( a_o \), respectively. The center-to-center separation distances from the emitter and receptor conductors to the reference conductor are \( d_E \) and \( d_R \), respectively. The center-to-center separation distance from the emitter conductor to the receptor conductor is \( d_{ER} \). Note that the insulating medium is assumed to be nonmagnetic (\( \mu = \mu_o \)).

![Diagram of three conductor line](image)

The \( 2 \times 2 \) per-unit-length (external) inductance matrix for the three conductor line is defined by

\[ l = \begin{bmatrix} I_E & I_m \\ I_m & I_R \end{bmatrix} \]
The self inductances of the emitter and receptor lines are given by

\[ l_E = \frac{\mu_0}{2\pi} \ln \left( \frac{d_E^2}{a_E a_o} \right) \]

\[ l_R = \frac{\mu_0}{2\pi} \ln \left( \frac{d_R^2}{a_R a_o} \right) \]

while the mutual inductance between the emitter and receptor lines is

\[ l_m = \frac{\mu_0}{2\pi} \ln \left( \frac{d_E d_R}{d_{ER} a_o} \right) \]

The per-unit-length capacitance matrix for the three conductor line is defined by

\[
\mathbf{c} = \begin{bmatrix}
  c_E + c_m & -c_m \\
  -c_m & c_R + c_m \\
\end{bmatrix} = \frac{\mu \varepsilon}{l} \mathbf{I}^{-1} = \frac{\mu \varepsilon}{l_E l_R - l_m^2} \begin{bmatrix}
  l_R & -l_m \\
  -l_m & l_E \\
\end{bmatrix}
\]

Equating the elements of the matrix terms on both sides of the equation gives

\[ c_m = \frac{\mu \varepsilon l_m}{l_E l_R - l_m^2} \]

\[ c_E = \frac{\mu \varepsilon (l_R - l_m)}{l_E l_R - l_m^2} \]

\[ c_R = \frac{\mu \varepsilon (l_E - l_m)}{l_E l_R - l_m^2} \]
The per-unit-length conductance matrix for the three conductor line is defined by

$$
\mathbf{g} = \begin{bmatrix}
    g_E + g_m & -g_m \\
    -g_m & g_R + g_m
\end{bmatrix} = \mu \sigma \mathbf{I}^{-1} = \frac{\mu \sigma}{l_E l_R - l_m^2} \begin{bmatrix}
    l_R & -l_m \\
    -l_m & l_E
\end{bmatrix}
$$

Equating the elements of the matrix terms on both sides of the equation gives

$$
g_m = \frac{\mu \sigma l_m}{l_E l_R - l_m^2} = \frac{\sigma}{\varepsilon} c_m
$$

$$
g_E = \frac{\mu \sigma (l_R - l_m)}{l_E l_R - l_m^2} = \frac{\sigma}{\varepsilon} c_E
$$

$$
g_R = \frac{\mu \sigma (l_E - l_m)}{l_E l_R - l_m^2} = \frac{\sigma}{\varepsilon} c_R
$$

Note that the per-unit-length self and mutual capacitance and conductance components for the three conductor line are identical in form except for the $\varepsilon$ and $\sigma$ terms in the numerators.

A special case of the three conductor line is that of a three conductor ribbon cable as shown below. The radius of the conductors is $a$ and the spacing between adjacent conductors is $d$. The center conductor of the ribbon cable is assumed to be the reference conductor.
Applying the general formulas for the self and mutual inductances of the three line to the special case of the three conductor ribbon cable (center conductor = reference conductor) gives

\begin{align*}
l_E &= \frac{\mu_0}{2\pi} \ln \left( \frac{d^2}{a_E a_o} \right) = \frac{\mu_0}{2\pi} \ln \left( \frac{d}{a} \right) = \frac{\mu_0}{\pi} \ln \left( \frac{d}{a} \right) \\
l_R &= l_E = \frac{\mu_0}{\pi} \ln \left( \frac{d}{a} \right) \\
l_m &= \frac{\mu_0}{2\pi} \ln \left( \frac{d_E d_R}{a_E a_o} \right) = \frac{\mu_0}{2\pi} \ln \left( \frac{d^2}{2da} \right) = \frac{\mu_0}{2\pi} \ln \left( \frac{d}{2a} \right)
\end{align*}

Another common example of a three conductor line is two round conductors over a conducting ground plane as shown below. The conducting ground plane is assumed to be the reference conductor in this configuration. The emitter and receptor conductors, of radii \(a_E\) and \(a_R\), respectively, are located at heights of \(h_E\) and \(h_R\) above the ground plane, respectively. The center-to-center spacing between the emitter conductor and the receptor conductor is \(s\).
The self and mutual inductances of the two conductors over ground configuration are

\[ I_E = \frac{\mu_0}{2\pi} \ln \left( \frac{2h_E}{a_E} \right) \]

\[ I_R = \frac{\mu_0}{2\pi} \ln \left( \frac{2h_R}{a_R} \right) \]

\[ I_m = \frac{\mu_0}{4\pi} \ln \left( 1 + \frac{4h_E h_R}{s^2} \right) \]

The configuration of two conductors within a cylindrical shield (reference conductor) is shown below. The radii of the emitter and receptor conductors is \(a_E\) and \(a_R\), respectively, while the inside radius of the cylindrical shield is \(a_{SH}\). The distances from the center of the shield to the centers of the emitter and the receptor conductors are \(d_E\) and \(d_R\), respectively. The angle formed by the lines connecting the wire centers to the shield center is defined as \(\theta\).
The self and mutual inductances for the three conductor shielded transmission line are

\[
I_k = \frac{\mu_0}{2\pi} \ln \left( \frac{a_{SH}^2 - d_k^2}{a_{SH}a_k} \right)
\]

\[
I_r = \frac{\mu_0}{2\pi} \ln \left( \frac{a_{SH}^2 - d_r^2}{a_{SH}a_r} \right)
\]

\[
I_m = \frac{\mu_0}{2\pi} \ln \left( \frac{d_r}{a_{SH}} \frac{(d_kd_r)^2 + a_{SH}^4 - 2d_kd_ra_{SH}^2 \cos \theta}{(d_kd_r)^2 + d_r^4 - 2d_kd_r^3 \cos \theta} \right)
\]

**Crosstalk Solution for a Lossless Line in a Homogeneous Media**

For a lossless three conductor line \((r = g = 0)\), the impedance and admittance matrices reduce to

\[
\hat{Z} = j\omega I
\]

\[
\hat{Y} = j\omega C
\]

and the product of the admittance and impedance matrices is

\[
\hat{Y}\hat{Z} = -\omega^2 Cl = -\omega^2 \mu \varepsilon I = -\beta^2 I
\]

which is a diagonal matrix. Thus, the transformation matrix required to diagonalize the admittance/impedance matrix product is simply the identity matrix.

\[
\hat{T} = I
\]

The matrix that relates the voltages and currents at the far end of the transmission line \((z = \infty)\) to the voltages and the currents at the near end \((z = 0)\) simplifies to the following terms.
This linear system can be solved for the currents at \( z = 0 \) and \( z = \varpi \). These currents can then be used to determine the near and far end voltages on the receptor line (the crosstalk responses).

\[
\hat{\mathbf{I}}(\varpi) = \left[ \begin{array}{c} \hat{I}_1(\varpi) \\ \hat{I}_2(\varpi) \end{array} \right] = \left[ \begin{array}{cc} \hat{\Phi}_{11} & \hat{\Phi}_{12} \\ \hat{\Phi}_{21} & \hat{\Phi}_{22} \end{array} \right] \left[ \begin{array}{c} \hat{V}_S - Z_S \hat{I}(0) \\ \hat{I}(0) \end{array} \right]
\]

\[
\hat{\Phi}_{11} = \frac{1}{2} \hat{\mathcal{R}}^{-1} \hat{\mathcal{I}}(e^{i\varpi} + e^{-i\varpi}) \hat{\mathcal{R}}^{-1} \hat{\mathcal{I}} = \cos \beta \varpi \mathbf{I}
\]

\[
\hat{\Phi}_{12} = -\frac{1}{2} \hat{\mathcal{R}}^{-1} \hat{\mathcal{I}}(e^{i\varpi} - e^{-i\varpi}) \hat{\mathcal{R}}^{-1} \hat{\mathcal{I}} = -j \omega \varpi \frac{\sin \beta \varpi}{\beta \varpi} \mathbf{l}
\]

\[
\hat{\Phi}_{21} = -\frac{1}{2} \hat{\mathcal{R}}(e^{i\varpi} - e^{-i\varpi}) \hat{\mathcal{R}}^{-1} \hat{\mathcal{I}} = -j \omega \varpi \frac{\sin \beta \varpi}{\beta \varpi} \mathbf{c}
\]

\[
\hat{\Phi}_{22} = \frac{1}{2} \hat{\mathcal{R}}(e^{i\varpi} + e^{-i\varpi}) \hat{\mathcal{R}}^{-1} = \cos \beta \varpi \mathbf{I}
\]

\[
\hat{V}_{NE} = \frac{S}{D} \left[ \frac{R_{NE}}{R_{NE} + R_{FE}} j \omega l_m \varpi \left( C + \frac{j 2 \pi \varpi / \lambda}{\sqrt{1 - k^2}} \alpha_{LE} S \right) \hat{I}_{EDC} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \varpi \left( C + \frac{j 2 \pi \varpi / \lambda}{\sqrt{1 - k^2}} \alpha_{LE} S \right) \hat{V}_{EDC} \right]
\]

\[
\hat{V}_{FE} = \frac{S}{D} \left[ -\frac{R_{FE}}{R_{NE} + R_{FE}} j \omega l_m \varpi \hat{I}_{EDC} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \varpi \hat{V}_{EDC} \right]
\]

where

\[
C = \cos \beta \varpi \quad S = \frac{\sin \beta \varpi}{\beta \varpi}
\]
The constant $k$ in the preceding equations is the coupling coefficient between the emitter and receptor lines and has a value between 0 and 1. The emitter and receptor circuits are weakly coupled if $k \ll 1$ and strongly coupled if $k \approx 1$. $\hat{V}_{E_{DC}}$ and $\hat{I}_{E_{DC}}$ represent the input voltage and current seen at the input of the emitter line under DC excitation where there is no coupling between the two lines. The terms $\tau_E$ and $\tau_R$ have units of time (resistance $\times$ capacitance or inductance/resistance) and are defined as the emitter and receptor line time constants. $Z_{oE}$ and $Z_{oR}$ can be defined as the coupled characteristic impedances of the emitter and receptor lines (the characteristic impedances of each line in the presence of the other line).
The “α” terms in the crosstalk voltage solutions represent the ratio of the respective termination to the coupled characteristic impedance at that location in the system. A termination is characterized as a low-impedance load if the value of α at that location of the circuit is less than one. A termination is characterized as a high-impedance load if the value of α is greater than one.

**Electrically Short, Weakly-Coupled Three Conductor Line**

If the three conductor line is electrically short (≪ₗ ≪ₗ), the general solution for the crosstalk voltages may be simplified given that

\[ C \approx 1 \quad S \approx 1 \]

If the emitter line is assumed to be weakly-coupled to the receptor line, then \( k \ll 1 \) such that

\[ D \approx 1 - \omega^2 \tau_E \tau_R + j \omega (\tau_E + \tau_R) = (1 + j \omega \tau_E)(1 + j \omega \tau_R) \]

and

\[
\hat{V}_{NE} \approx \frac{1}{D} \left[ \frac{R_{NE}}{R_{NE} + R_{FE}} j \omega l_m \phi \hat{I}_{EDC} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \phi \hat{V}_{EDC} \right] \\
\hat{V}_{FE} \approx \frac{1}{D} \left[ -\frac{R_{FE}}{R_{NE} + R_{FE}} j \omega l_m \phi \hat{I}_{EDC} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \phi \hat{V}_{EDC} \right]
\]

The time constants \( \tau_E \) and \( \tau_R \) define critical frequencies associated with the emitter line and the receptor line.

\[ f_E = \frac{1}{\tau_E} \quad f_R = \frac{1}{\tau_R} \]
If we assume that the operating frequency is much less than both \( f_E \) and \( f_R \), then \( D \approx 1 \) and the crosstalk voltages reduce to

\[
\hat{V}_{NE} \approx \left[ \frac{R_{NE}}{R_{NE} + R_{FE}} j \omega l_m \varpi \hat{I}_{EDC} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \varpi \hat{V}_{EDC} \right]
\]

\[
\hat{V}_{FE} \approx \left[ -\frac{R_{FE}}{R_{NE} + R_{FE}} j \omega l_m \varpi \hat{I}_{EDC} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \varpi \hat{V}_{EDC} \right]
\]

The crosstalk responses on the electrically short, weakly-coupled line are clearly stated in terms of the mutual inductance and mutual capacitance between the emitter and receptor lines. The terms involving the mutual inductance define *inductive coupling* between the lines while the terms involving the mutual capacitance define the *capacitive coupling* between the lines. A simple equivalent circuit defining the crosstalk responses is shown below.

\[ j \omega l_m \varpi \hat{I}_{EDC} \]

\[ R_{NE} \]

\[ \hat{V}_{NE} \]

\[ j \omega c_m \varpi \hat{V}_{EDC} \]

\[ R_{FE} \]

\[ \hat{V}_{FE} \]

The coupling mechanism which dominates the crosstalk response will depend on the line terminations. For low-impedance loads with \( \alpha \ll 1 \), we expect low voltages and high currents, which corresponds to effective inductive coupling. For high-impedance loads with \( \alpha \gg 1 \), we expect high voltages and low currents, which corresponds to effective capacitive coupling.
The near and far end crosstalk responses can be written as the sum of inductively-coupled and capacitively-coupled terms as

\[
\hat{V}_{NE} \approx \left[ \frac{R_{NE}}{R_{NE} + R_{FE}} j \omega l_m \zeta \hat{I}_{E_{DC}} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \zeta \hat{V}_{E_{DC}} \right] = \hat{V}^{IND}_{NE} + \hat{V}^{CAP}_{NE}
\]

\[
\hat{V}_{FE} \approx \left[ -\frac{R_{FE}}{R_{NE} + R_{FE}} j \omega l_m \zeta \hat{I}_{E_{DC}} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} j \omega c_m \zeta \hat{V}_{E_{DC}} \right] = \hat{V}^{IND}_{FE} + \hat{V}^{CAP}_{FE}
\]

Note that the capacitively-coupled responses at the ends of the receptor line are identical (same magnitude, same sign). The magnitude of the inductively-coupled responses at the ends of the receptor line are dependent on the termination at either end. The inductively-coupled responses at the ends of the receptor line are of opposite sign.

The ratio of the capacitively-coupled response to the inductively-coupled response at each end of the receptor line gives

\[
\frac{\hat{V}^{CAP}_{NE}}{\hat{V}^{IND}_{NE}} = \frac{c_m}{l_m} \frac{R_{L}}{R_{FE}} \quad \frac{\hat{V}^{CAP}_{FE}}{\hat{V}^{IND}_{FE}} = -\frac{c_m}{l_m} \frac{R_{L}}{R_{NE}}
\]

From the definition of the coupling coefficient \(k\) and the coupled characteristic impedances for the emitter and receptor lines (\(Z_{oE}\) and \(Z_{oR}\)), the ratio of mutual inductance to the mutual capacitance can be shown to be equivalent to the product of the coupled characteristic impedances.

\[
l_m = \sqrt{\frac{l_E}{l_R}} = \sqrt{\frac{Z_{oE}}{Z_{oR}}} \quad c_m = \sqrt{\frac{l_E}{l_R}} = \sqrt{\frac{Z_{oE}}{Z_{oR}}}
\]

Inserting this result into the voltage response ratios, and taking the magnitude of both sides gives

\[
\left| \frac{\hat{V}^{CAP}_{NE}}{\hat{V}^{IND}_{NE}} \right| = \frac{R_L R_{FE}}{Z_{oE} Z_{oR}} = \alpha_{LE} \alpha_{LR} \quad \left| \frac{\hat{V}^{CAP}_{FE}}{\hat{V}^{IND}_{FE}} \right| = \frac{R_L R_{NE}}{Z_{oE} Z_{oR}} = \alpha_{LE} \alpha_{SR}
\]
Based on the previous equations, we may easily determine which coupling mechanism dominates the crosstalk response at either end of the receptor line.

Near-end receptor response

\[ |\hat{V}_{NE}^{CAP}| > |\hat{V}_{NE}^{IND}| \quad \text{if} \quad \alpha_{LE} \alpha_{LR} > 1 \]

\[ |\hat{V}_{NE}^{CAP}| < |\hat{V}_{NE}^{IND}| \quad \text{if} \quad \alpha_{LE} \alpha_{LR} < 1 \]

Far-end receptor response

\[ |\hat{V}_{FE}^{CAP}| > |\hat{V}_{FE}^{IND}| \quad \text{if} \quad \alpha_{LE} \alpha_{SR} > 1 \]

\[ |\hat{V}_{FE}^{CAP}| < |\hat{V}_{FE}^{IND}| \quad \text{if} \quad \alpha_{LE} \alpha_{SR} < 1 \]

Note that the capacitive coupling response is dominant when the respective terminations are high-impedance loads while the inductive coupling response is dominant when the respective terminations are low-impedance loads.

The crosstalk responses at either end of the receptor line can be written as transfer functions with respect to the source voltage by inserting

\[ \hat{V}_{E_{dc}} = \frac{R_L}{R_S + R_L} \hat{V}_S \quad \hat{i}_{E_{dc}} = \frac{1}{R_S + R_L} \hat{i}_S \]

into the expressions for the voltage responses. This gives

\[ \frac{\hat{V}_{NE}}{\hat{V}_S} \approx j\omega \left[ \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{L_m}{R_S + R_L} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{R_L C_m}{R_S + R_L} \right] \]

\[ \frac{\hat{V}_{FE}}{\hat{V}_S} \approx j\omega \left[ -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{L_m}{R_S + R_L} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{R_L C_m}{R_S + R_L} \right] \]

where \( L_m = \mathcal{L} l_m \) and \( C_m = \mathcal{L} c_m \).
The near end and far end transfer functions can be written as a superposition of inductive and capacitive coupling terms such that

\[
\hat{V}_{NE} = \hat{V}^{\text{IND}}_{NE} + \hat{V}^{\text{CAP}}_{NE} = j \omega \left[ M_{\text{IND}}^{\text{NE}} + M_{\text{CAP}}^{\text{NE}} \right]
\]

\[
\hat{V}_{FE} = \hat{V}^{\text{IND}}_{FE} + \hat{V}^{\text{CAP}}_{FE} = j \omega \left[ M_{\text{IND}}^{\text{FE}} + M_{\text{CAP}}^{\text{FE}} \right]
\]

where

\[
M_{\text{IND}}^{\text{NE}} = \frac{R_{\text{NE}}}{R_{\text{NE}} + R_{\text{FE}}} \frac{L_m}{R_S + R_L}
\]

\[
M_{\text{IND}}^{\text{FE}} = -\frac{R_{\text{FE}}}{R_{\text{NE}} + R_{\text{FE}}} \frac{L_m}{R_S + R_L}
\]

\[
M_{\text{CAP}}^{\text{NE}} = M_{\text{CAP}}^{\text{FE}} = \frac{R_{\text{NE}} R_{\text{FE}}}{R_{\text{NE}} + R_{\text{FE}}} \frac{R_L C_m}{R_S + R_L}
\]

Note that the overall crosstalk transfer functions including inductive and capacitive coupling terms are directly proportional to \( s = j \omega \) (like the frequency response of an ideal inductor). Thus, these transfer functions (together or individually) may be represented on a Bode plot as a straight lines with a +20 dB/decade slope. The crosstalk transfer functions have the same frequency dependence when dominated by inductive coupling, when dominated by capacitive coupling, or when both coupling mechanisms provide a significant combination.
COMMON-IMPEDANCE COUPLING

The crosstalk transfer functions developed in the preceding analysis were determined under the assumption of a lossless three conductor transmission line. The basic form of the crosstalk transfer functions (that of an ideal inductor) predicts that the inductive and capacitive coupling between the emitter and receptor lines diminishes as the frequency approaches zero. However, when the resistive losses associated with the conductors of the transmission line are included, a different coupling mechanism known as common-impedance coupling produces significant coupling at low frequencies.

Consider the electrically short three conductor line shown below where the resistance of the transmission line reference conductor has been included in the model. The resistances of the emitter and receptor conductors are typically very small in comparison to the termination impedances and can be neglected.

At low frequencies, the electromagnetic coupling between the emitter and receptor lines can be assumed to be negligible such that the current in the receptor circuit is negligible. The current in the emitter circuit flows through the reference conductor producing a voltage drop of

\[
\hat{V}_o = \hat{I}_E R_o \approx \frac{\hat{V}_S}{R_s + R_L + R_o} R_o \approx \frac{\hat{V}_S}{R_s + R_L} R_o
\]
The voltage drop across the reference conductor produces a voltage across
the series combination of $R_{NE}$ and $R_{FE}$. The responses seen in the near end
and far end terminations of the receptor circuit due to common-impedance
coupling are

$$\hat{V}_{NE}^{CI} = \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{R_o}{R_S + R_L} \hat{V}_S$$

$$\hat{V}_{FE}^{CI} = -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{R_o}{R_S + R_L} \hat{V}_S$$

Note that the common-impedance coupling responses are frequency
independent. The common-impedance coupling responses can be written
in the form of transfer functions as

$$\hat{V}_{NE}^{CI} = M_{NE}^{CI} = \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{R_o}{R_S + R_L}$$

$$\hat{V}_{FE}^{CI} = M_{FE}^{CI} = -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{R_o}{R_S + R_L}$$

Combining the common-impedance coupling terms with the inductive and
capacitive coupling terms gives the overall crosstalk transfer functions.

$$\hat{V}_{NE} = \frac{\hat{V}_{NE}^{IND}}{\hat{V}_S} + \frac{\hat{V}_{NE}^{CAP}}{\hat{V}_S} + \frac{\hat{V}_{NE}^{CI}}{\hat{V}_S} = j\omega \left[ M_{NE}^{IND} + M_{NE}^{CAP} \right] + M_{NE}^{CI}$$

$$\hat{V}_{FE} = \frac{\hat{V}_{FE}^{IND}}{\hat{V}_S} + \frac{\hat{V}_{FE}^{CAP}}{\hat{V}_S} + \frac{\hat{V}_{FE}^{CI}}{\hat{V}_S} = j\omega \left[ M_{FE}^{IND} + M_{FE}^{CAP} \right] + M_{FE}^{CI}$$

The common-impedance terms in the crosstalk transfer functions yield a
low-frequency asymptote in the Bode plots of the transfer function
magnitudes.
Note that, in general, the common-impedance coupling asymptotes for the near end and the far end responses on the receptor line are different levels since these asymptotes depend on the terminations $R_{NE}$ and $R_{FE}$. The Bode plot approximations for the near end and far end crosstalk transfer functions are only valid at sufficiently low frequencies where the transmission line length is electrically short. At higher frequencies, the resonant properties of the conductors produce wide variation in the magnitudes of the transfer functions.

The frequencies at which common-impedance coupling and electromagnetic coupling asymptotes meet represents the frequency where to two coupling levels are equal. Solving for these frequencies yields

\[
f_{NE} = \frac{M^{CI}_{NE}}{2\pi \left[ M^{IND}_{NE} + M^{CAP}_{NE} \right]} \]

\[
f_{FE} = \frac{M^{CI}_{FE}}{2\pi \left[ M^{IND}_{FE} + M^{CAP}_{FE} \right]} \]
Example

Determine the Bode plot approximations of the near end and far end crosstalk transfer functions for a three conductor ribbon cable with #28 gauge (7×36) stranded copper wires spaced by $d = 50$ mils. The ribbon cable is 0.7m long and the terminations are defined by $R_L = R_{NE} = R_{FE} = 50$ $\Omega$ and $R_S = 0$ $\Omega$.

#28 gauge (7×36) $\Rightarrow a = 7.5$ mils $= 0.1905$ mm

$$l_E = l_R = \frac{\mu_0}{\pi} \ln \left( \frac{d}{a} \right) = (4 \times 10^{-7}) \ln (50/7.5) = 0.759 \, \mu\text{H/m}$$

$$l_m = \frac{\mu_0}{2\pi} \ln \left( \frac{d}{2a} \right) = (2 \times 10^{-7}) \ln (50/15) = 0.241 \, \mu\text{H/m}$$

$$L_m = l_m \varnothing = (0.241 \, \mu\text{H/m})(0.7 \text{ m}) = 0.169 \, \mu\text{H}$$

$$c_R = c_E = \frac{\mu_0 \epsilon_0 (l_E - l_m)}{l_E l_R - l_m^2} = 11.1 \, \text{pF/m}$$

$$c_m = \frac{\mu_0 \epsilon_0 l_m}{l_E l_R - l_m^2} = 5.17 \, \text{pF/m}$$

$$C_m = c_m \varnothing = (5.17 \, \text{pF/m})(0.7 \text{ m}) = 3.62 \, \text{pF}$$

$$Z_{oE} = Z_{oR} = \sqrt{\frac{l_E}{c_E + c_m}} = 216 \, \Omega$$
\[
\alpha_{\text{SE}} = \frac{R_S}{Z_{oE}} = 0
\]
\[
\alpha_{\text{LE}} = \frac{R_L}{Z_{oE}} = \alpha_{\text{SR}} = \frac{R_{\text{NE}}}{Z_{oR}} = \alpha_{\text{LR}} = \frac{R_{\text{FE}}}{Z_{oR}} = \frac{50}{216} = 0.231
\]

Near-end receptor response

\[\alpha_{\text{LE}} \alpha_{\text{LR}} < 1 \quad \text{(inductive coupling dominates)}\]

Far-end receptor response

\[\alpha_{\text{LE}} \alpha_{\text{SR}} < 1 \quad \text{(inductive coupling dominates)}\]

#36 gauge wire \( \Rightarrow \) \( a = 2.5 \) mils = 0.0635 mm

\[
R_{\text{str}} = \frac{l}{\sigma A} = 0.7 / [(5.8 \times 10^7) \pi (0.0635 \times 10^{-4})^2]
\]

\[= 0.953 \ \Omega\]

\[R_o = R_{\text{str}} / (\# \text{ of strands}) = 0.953 \ \Omega / 7 = 0.136 \ \Omega\]

\[
M_{\text{IND}}^{\text{NE}} = \frac{R_{\text{NE}}}{R_{\text{NE}} + R_{\text{FE}}} \frac{L_m}{R_S + R_L} = 1.69 \times 10^{-9}
\]

\[
M_{\text{IND}}^{\text{FE}} = -\frac{R_{\text{FE}}}{R_{\text{NE}} + R_{\text{FE}}} \frac{L_m}{R_S + R_L} = -1.69 \times 10^{-9}
\]

\[
M_{\text{CAP}}^{\text{NE}} = M_{\text{CAP}}^{\text{FE}} = \frac{R_{\text{NE}} R_{\text{FE}}}{R_{\text{NE}} + R_{\text{FE}}} \frac{R_L C_m}{R_S + R_L} = 90.5 \times 10^{-12}
\]
\[ M_{NE}^{CI} = \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{R_o}{R_s + R_L} = 1.36 \times 10^{-3} \]

\[ M_{FE}^{CI} = -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{R_o}{R_s + R_L} = -1.36 \times 10^{-3} \]

\[ 20 \log_{10} |M_{NE}^{CI}| = 20 \log_{10} |M_{FE}^{CI}| = -57.33 \text{ dB} \]

\[ f_{NE} = \frac{M_{NE}^{CI}}{2\pi [M_{NE}^{IND} + M_{NE}^{CAP}]} = 122 \text{ kHz} \]

\[ f_{FE} = \frac{M_{FE}^{CI}}{2\pi [M_{FE}^{IND} + M_{FE}^{CAP}]} = 135 \text{ kHz} \]

\[ \frac{\hat{V}_{NE}}{\hat{V}_s} = M_{NE}^{CI} + j \omega [M_{NE}^{IND} + M_{NE}^{CAP}] \]

\[ = 1.36 \times 10^{-3} + s(1.78 \times 10^{-9}) \]

\[ \frac{\hat{V}_{FE}}{\hat{V}_s} = j \omega [M_{FE}^{IND} + M_{FE}^{CAP}] + M_{FE}^{CI} \]

\[ = -1.36 \times 10^{-3} - s(1.60 \times 10^{-9}) \]
Example

Determine the Bode plot approximations of the near end and far end crosstalk transfer functions for the same three conductor ribbon cable with \( R_L = R_{NE} = R_{FE} = 1000 \, \Omega \) and \( R_S = 0 \, \Omega \).

\[
\alpha_{SE} = \frac{R_S}{Z_{oE}} = 0
\]

\[
\alpha_{LE} = \frac{R_L}{Z_{oE}} = \alpha_{SR} = \frac{R_{NE}}{Z_{oR}} = \alpha_{LR} = \frac{R_{FE}}{Z_{oR}} = \frac{1000}{216} = 4.63
\]

Near-end receptor response

\[ \alpha_{LE} \alpha_{LR} > 1 \quad \text{(capacitive coupling dominates)} \]

Far-end receptor response

\[ \alpha_{LE} \alpha_{SR} > 1 \quad \text{(capacitive coupling dominates)} \]

\[
M_{NE}^{IND} = \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{L_m}{R_S + R_L} = 8.45 \times 10^{-11}
\]

\[
M_{FE}^{IND} = -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{L_m}{R_S + R_L} = -8.45 \times 10^{-11}
\]

\[
M_{NE}^{CAP} = M_{FE}^{CAP} = \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{R_L C_m}{R_S + R_L} = 1.81 \times 10^{-9}
\]

\[
M_{NE}^{CI} = \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{R_o}{R_S + R_L} = 6.80 \times 10^{-5}
\]

\[
M_{FE}^{CI} = -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{R_o}{R_S + R_L} = -6.80 \times 10^{-5}
\]
\[ 20 \log_{10} |M_{NE}^{CI}| = 20 \log_{10} |M_{FE}^{CI}| = -83.35 \text{ dB} \]

\[ f_{NE} = \frac{M_{NE}^{CI}}{2\pi [M_{NE}^{IND} + M_{NE}^{CAP}]} = 5.71 \text{ kHz} \]

\[ f_{FE} = \frac{M_{FE}^{CI}}{2\pi [M_{FE}^{IND} + M_{FE}^{CAP}]} = 6.27 \text{ kHz} \]

\[ \frac{\hat{V}_{NE}}{\hat{V}_S} = M_{NE}^{CI} + j \omega [M_{NE}^{IND} + M_{NE}^{CAP}] \]

\[ = 6.80 \times 10^{-5} + s(1.78 \times 10^{-9}) \]

\[ \frac{\hat{V}_{FE}}{\hat{V}_S} = M_{FE}^{CI} + j \omega [M_{FE}^{IND} + M_{FE}^{CAP}] \]

\[ = -6.80 \times 10^{-5} + s(1.89 \times 10^{-9}) \]
**TIME-DOMAIN CROSSTALK**

The frequency-domain near-end and far-end crosstalk responses for a lossless, electrically short, weakly-coupled three-conductor transmission line (neglecting the low frequency effect of common impedance coupling) can be written as

\[
\hat{V}_{NE} \approx j \omega \left[ \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{L_m}{R_s + R_L} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{R_L C_m}{R_s + R_L} \right] \hat{V}_S
\]

\[
= j \omega \left[ M_{NE}^{IND} + M_{NE}^{CAP} \right] \hat{V}_S
\]

\[
= j \omega M_{NE} \hat{V}_S
\]

\[
\hat{V}_{FE} \approx j \omega \left[ -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{L_m}{R_s + R_L} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{R_L C_m}{R_s + R_L} \right] \hat{V}_S
\]

\[
= j \omega \left[ M_{FE}^{IND} + M_{FE}^{CAP} \right] \hat{V}_S
\]

\[
= j \omega M_{FE} \hat{V}_S
\]

The multiplier “\(j\omega\)” in the frequency-domain translates into a derivative with respect to time in the time-domain.

\[
\frac{d}{dt} \quad \leftrightarrow \quad j \omega
\]

Thus, the instantaneous crosstalk responses can be written concisely as

\[
v_{NE}(t) = M_{NE} \frac{dv_S(t)}{dt}
\]

\[
v_{FE}(t) = M_{FE} \frac{dv_S(t)}{dt}
\]
The time-domain equivalent circuit describing the instantaneous crosstalk response is obtained by transforming the previously defined phasor crosstalk equivalent circuit from the frequency-domain to the time-domain as shown below.

**Frequency-Domain (phasor crosstalk responses)**

\[
\begin{align*}
\hat{V}_{NE} & = j\omega l_m \hat{I}_{E,DC} \\
\hat{V}_{FE} & = j\omega c_m \hat{V}_{E,DC} \\
\hat{V}_{E,DC} & = \frac{R_L}{R_S+R_L} \hat{V}_S \\
\hat{I}_{E,DC} & = \frac{1}{R_S+R_L} \hat{V}_S
\end{align*}
\]

**Time-Domain (instantaneous crosstalk responses)**

\[
\begin{align*}
\frac{L_m}{dt} \frac{di_E(t)}{dt} & = R_{NE} v_{NE}(t) + C_m \frac{dv_E(t)}{dt} \\
v_E(t) & = \frac{R_L}{R_S+R_L} v_S(t) \\
i_E(t) & = \frac{1}{R_S+R_L} v_S(t)
\end{align*}
\]

The instantaneous near-end and far-end crosstalk responses are obtained by analyzing the time-domain equivalent circuit using superposition for the two sources, along with voltage division and current division. The resulting near end and far end instantaneous crosstalk responses are
where

\[ L_m \frac{d i_E(t)}{dt} = \frac{L_m}{R_s + R_L} \frac{d v_S(t)}{dt} \]

\[ C_m \frac{d v_E(t)}{dt} = \frac{C_m R_L}{R_s + R_L} \frac{d v_S(t)}{dt} \]

The instantaneous crosstalk responses written in terms of the emitter line source voltage are

\[ v_{NE}(t) = \frac{R_{NE}}{R_{NE} + R_{FE}} \left( L_m \frac{d i_E(t)}{dt} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{C_m R_L}{R_s + R_L} \frac{d v_S(t)}{dt} \right) \]

\[ v_{FE}(t) = -\frac{R_{FE}}{R_{NE} + R_{FE}} \left( L_m \frac{d i_E(t)}{dt} + \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{C_m R_L}{R_s + R_L} \frac{d v_S(t)}{dt} \right) \]

Note that the near-end and far-end crosstalk responses are scaled derivatives of the source voltage. Thus, the time-domain crosstalk responses are most significant when the source voltage is changing rapidly in time. The amplitudes of the crosstalk responses are proportional to the derivative of the source voltage and the size of the respective mutual coupling term.

\[ M_{NE} = M_{NE}^{IND} + M_{NE}^{CAP} \]

\[ M_{FE} = M_{FE}^{IND} + M_{FE}^{CAP} \]

Note that the near-end mutual coupling term \( M_{NE} \) is always positive but the far-end mutual coupling term \( M_{FE} \) may be positive or negative.
The instantaneous near-end and far-end crosstalk responses to a piecewise linear representation of a clock signal source voltage are shown below.

Note that the sign of the near-end crosstalk response is known ($M_{NE}$ is positive) but the sign of the far-end response depends on the sign of the far-end mutual coupling term $M_{FE}$.
Determine the peak values of the near-end and far-end crosstalk responses for the previously considered three conductor ribbon cable [#28 gauge (7×36) stranded copper wires spaced by \(d = 50\) mils, \(L = 0.7\) m, \(R_L = R_{NE} = R_{FE} = 50\)  \(\Omega\), \(R_S = 0\) \(\Omega\)] given a 10 MHz clock signal [5 volt amplitude, 50 \% duty cycle, 10 ns rise/fall times].

From previous results ...

**Near-end receptor response**

\[\alpha_{LE}\alpha_{LR} < 1 \quad \text{(inductive coupling dominates)}\]

**Far-end receptor response**

\[\alpha_{LE}\alpha_{SR} < 1 \quad \text{(inductive coupling dominates)}\]

\[
M_{NE}^{IND} = \frac{R_{NE}}{R_{NE} + R_{FE}} \frac{L_m}{R_S + R_L} = 1.69 \times 10^{-9} \text{ s}
\]

\[
M_{FE}^{IND} = -\frac{R_{FE}}{R_{NE} + R_{FE}} \frac{L_m}{R_S + R_L} = -1.69 \times 10^{-9} \text{ s}
\]

\[
M_{NE}^{CAP} = M_{FE}^{CAP} = \frac{R_{NE} R_{FE}}{R_{NE} + R_{FE}} \frac{R_L C_m}{R_S + R_L} = 90.5 \times 10^{-12} \text{ s}
\]

The near-end and far-end coupling coefficients are

\[
M_{NE} = M_{NE}^{IND} + M_{NE}^{CAP} = 1.78 \times 10^{-9} \text{ s}
\]

\[
M_{FE} = M_{FE}^{IND} + M_{FE}^{CAP} = -1.60 \times 10^{-9} \text{ s}
\]

The magnitude of the near-end and far-end crosstalk responses at the clock pulse transitions are
\[ |v_{NE}(t)| = |M_{NE}| \left| \frac{dv_s(t)}{dt} \right| \]

\[ = |M_{NE}| \frac{A}{\tau_r} = 1.78 \times 10^{-9} \frac{5}{10 \times 10^{-9}} = 0.89 \text{ V} \]

\[ |v_{PE}(t)| = |M_{PE}| \left| \frac{dv_s(t)}{dt} \right| \]

\[ = |M_{PE}| \frac{A}{\tau_r} = 1.60 \times 10^{-9} \frac{5}{10 \times 10^{-9}} = 0.80 \text{ V} \]