Wire Antennas

*Electrical Size of an Antenna* - the physical dimensions of the antenna defined relative to wavelength.

*Electrically small antenna* - the dimensions of the antenna are small relative to wavelength.

*Electrically large antenna* - the dimensions of the antenna are large relative to wavelength.

**Example**  Consider a dipole antenna of length $L = 1\text{m}$. Determine the electrical length of the dipole at $f = 3 \text{ MHz}$ and $f = 30 \text{ GHz}$.

\[
\begin{array}{ccc}
L = 1\text{m} & L = \lambda/100 & L = 100\lambda \\
\text{Electrically small} & f = 3 \text{ MHz} & f = 30 \text{ GHz} \\
(\lambda = 100\text{m}) & (\lambda = 0.01\text{m}) & \\
\text{Electrically large} & & \\
\end{array}
\]
We assume that the axial current along the infinitesimal dipole is uniform. With \(a \ll \lambda\), we may assume that any circumferential currents are negligible and treat the dipole as a current filament.

The infinitesimal dipole with a constant current along its length is a non-physical antenna. However, the infinitesimal dipole approximates several physically realizable antennas.
Capacitor-plate antenna (top-hat-loaded antenna)

The “capacitor plates” can be actual conductors or simply the wire equivalent. The fields radiated by the radial currents tend to cancel each other in the far field so that the far fields of the capacitor plate antenna can be approximated by the infinitesimal dipole.

Transmission line loaded antenna

If we assume that $L \approx \lambda/4$, then the current along the antenna resembles that of a half-wave dipole.
Inverted-L antenna

Using image theory, the inverted-L antenna is equivalent to the transmission line loaded antenna.

Based on the current distributions on these antennas, the far fields of the capacitor plate antenna, the transmission line loaded antenna and the inverted-L antenna can all be approximated by the far fields of the infinitesimal dipole.
To determine the fields radiated by the infinitesimal dipole, we first determine the magnetic vector potential $A$ due to the given electric current source $J (M = 0, F = 0)$.

$$A(r) = \frac{\mu}{4\pi} \int \int \int J(r') \frac{e^{-jk|r-r'|}}{|r-r'|} \, dv'$$

$$J(r') \rightarrow I(z') = I_o a_z$$

$$dv' \rightarrow dl' = dz' \quad \left(-\frac{\Delta l}{2} \leq z' \leq \frac{\Delta l}{2}\right)$$

$r' \approx 0$ (point source)

$$|r-r'| \approx |r| = r$$

$$A(r) \approx \frac{\mu}{4\pi} \int_{-\Delta l/2}^{\Delta l/2} I_o a_z \left(\frac{e^{-jk r}}{r}\right) dz'$$

$$= \frac{\mu I_o}{4\pi r} e^{-jkr} a_z \int_{-\Delta l/2}^{\Delta l/2} dz'$$

$$= \frac{\mu I_o \Delta l}{4\pi r} e^{-jkr} a_z$$

$$= A_z(r) a_z$$

The infinitesimal dipole magnetic vector potential given in the previous equation is a rectangular coordinate vector with the magnitude defined in terms of spherical coordinates. The rectangular coordinate vector can be transformed into spherical coordinates using the standard coordinate transformation.

$$A_x, A_y, A_z \rightarrow A_r, A_\theta, A_\phi$$
The total magnetic vector potential may then be written in vector form as

\[ \mathbf{A} = \mathbf{A}_r + \mathbf{A}_\theta + \mathbf{A}_\phi \]

\[ A_r = A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta \]

\[ A_\theta = A_x \cos \theta \cos \phi + A_y \cos \theta \sin \phi - A_z \sin \theta \]

\[ A_\phi = -A_x \sin \phi + A_y \cos \phi \]

\[ A_r = \frac{\mu I_0 \Delta l}{4 \pi r} e^{-jkr} \cos \theta = A_r(r, \theta) \]

\[ A_\theta = -\frac{\mu I_0 \Delta l}{4 \pi r} e^{-jkr} \sin \theta = A_\theta(r, \theta) \]

\[ A_\phi = 0 \]

The total magnetic vector potential may then be written in vector form as

\[ \mathbf{A}(r, \theta) = A_r(r, \theta) \mathbf{a}_r + A_\theta(r, \theta) \mathbf{a}_\theta \]

\[ = \frac{\mu I_0 \Delta l}{4 \pi r} e^{-jkr} (\cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta) \]

Because of the true point source nature of the infinitesimal dipole (\( \Delta l \approx \lambda/50 \)), the equation above for the magnetic vector potential of the infinitesimal dipole is valid everywhere. We may use this expression for \( \mathbf{A} \) to determine both near fields and far fields.

The radiated fields of the infinitesimal dipole are found by differentiating the magnetic vector potential.

\[ \mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \]
\[ \nabla \times A = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] a_r \\
+ \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial A_\phi}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] a_\theta \\
+ \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] a_\phi \]

\[ H = \frac{1}{\mu} \nabla \times A = \frac{1}{\mu r} \left[ -\frac{\mu I_o \Delta l}{4 \pi} \sin \theta \frac{\partial}{\partial r} (e^{-jkr}) \\
- \frac{\mu I_o \Delta l}{4 \pi r} e^{-jkr} \frac{\partial}{\partial \theta} (\cos \theta) \right] a_\phi \]

\[ = \frac{I_o \Delta l}{4 \pi r} \left[ -\sin \theta (-jke^{-jkr}) - \frac{e^{-jkr}}{r} (-\sin \theta) \right] a_\phi \]

\[ = \frac{jkI_o \Delta l}{4 \pi} \sin \theta \left[ \frac{1}{r} + \frac{1}{jkr^2} \right] e^{-jkr} a_\phi \]

\[ = \frac{I_o \Delta l}{4 \pi} \sin \theta \left[ \frac{jk}{r} + \frac{1}{r^2} \right] e^{-jkr} a_\phi \]

\[ = H_\phi(r, \theta) a_\phi \]
The electric field is found using either potential theory or Maxwell’s equations.

**Potential Theory**

\[ E = -j \omega A - \nabla \phi_e \]

\[ = -j \omega A - \frac{j}{\omega \mu \epsilon} \nabla (\nabla \cdot A) \]

\[ = -j \omega \left[ A + \frac{1}{k^2} \nabla (\nabla \cdot A) \right] \]

**Maxwell’s Equations** \((J = 0 \text{ away from the source})\)

\[ \nabla \times H = j \omega \epsilon E + \mathbf{J} \]

\[ \implies E = \frac{1}{j \omega \epsilon} \nabla \times H \]

Note that electric field expression in terms of potentials requires two levels of differentiation while the Maxwell’s equations equation requires only one level of differentiation. Thus, using Maxwell’s equations, we find

\[ \nabla \times H = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (H_\phi \sin \theta) - \frac{\partial H_\theta}{\partial \phi} \right] a_r \]

\[ + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial}{\partial r} (r H_\phi) \right] a_\theta \]

\[ + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{\partial \theta} \right] a_\phi \]
\[
E = \frac{1}{j \omega \epsilon} \left( \frac{j k I_o \Delta l}{4 \pi} \right) \left\{ \frac{1}{r \sin \theta} \left[ \frac{1}{r} + \frac{1}{j kr^2} \right] e^{-j kr} \frac{\partial}{\partial \theta} (\sin^2 \theta) a_r \right. \\
- \frac{1}{r} \sin \theta \frac{\partial}{\partial r} \left[ \left( 1 + \frac{1}{j kr} \right) e^{-j kr} \right] a_\theta \left\}
\]

\[
e = \frac{\eta I_o \Delta l}{4 \pi} \left\{ \frac{1}{r \sin \theta} \left[ \frac{1}{r} + \frac{1}{j kr^2} \right] e^{-j kr} (2 \sin \theta \cos \theta) a_r \\
- \frac{1}{r} \sin \theta \left[ -j ke^{-j kr} + \left( \frac{1}{j kr} \right) (-j ke^{-j kr}) + \left( e^{-j kr} \right) \left( -\frac{1}{j kr^2} \right) \right] a_\theta \right\}
\]

\[
e = \frac{\eta I_o \Delta l}{4 \pi} \left\{ 2 \cos \theta \left[ \frac{1}{r^2} + \frac{1}{j kr^3} \right] e^{-j kr} a_r \\
+ \sin \theta \left[ \frac{j k}{r} + \frac{1}{r^2} + \frac{1}{j kr^3} \right] e^{-j kr} a_\theta \right\}
\]

\[
e = E_r(r, \theta) a_r + E_\theta(r, \theta) a_\theta
\]

\[
E_r = \eta \frac{k I_o \Delta l}{2 \pi r} \cos \theta \left[ \frac{1}{kr} - \frac{j}{(kr)^2} \right] e^{-j kr}
\]

\[
E_\theta = \eta \frac{k I_o \Delta l}{4 \pi r} \sin \theta \left[ j + \frac{1}{kr} - \frac{j}{(kr)^2} \right] e^{-j kr}
\]

\[
H_\phi = \frac{k I_o \Delta l}{4 \pi r} \sin \theta \left[ j + \frac{1}{kr} \right] e^{-j kr}
\]

fields radiated by an infinitesimal dipole
Field Regions of the Infinitesimal Dipole

We may separate the fields of the infinitesimal dipole into the three standard regions:

① Reactive near field \( kr << 1 \)
② Radiating near field \( kr > 1 \)
③ Far field \( kr >> 1 \)

Considering the bracketed terms \([\ ]\) in the radiated field expressions for the infinitesimal dipole ...

① Reactive near field \( (kr << 1) \) \((kr)^2\) terms dominate
② Radiating near field \( (kr > 1) \) constant terms dominate if present otherwise, \((kr)^{-1}\) terms dominate
③ Far field \( (kr >> 1) \) constant terms dominate
Reactive near field \([ kr << 1 \text{ or } r << \lambda/2\pi \)\]

When \( kr << 1 \), the terms which vary inversely with the highest power of \( kr \) are dominant. Thus, the near field of the infinitesimal dipole is given by

\[
E_r \approx -j\eta \frac{I_o \Delta l e^{-jkr}}{2\pi kr^3} \cos\theta
\]

\[
E_\theta \approx -j\eta \frac{I_o \Delta l e^{-jkr}}{4\pi kr^3} \sin\theta
\]

\[
H_\phi \approx \frac{I_o \Delta l e^{-jkr}}{4\pi r^2} \sin\theta
\]

Infinitesimal dipole near fields

Note the 90° phase difference between the electric field components and the magnetic field component (these components are in phase quadrature) which indicates reactive power (stored energy, not radiation). If we investigate the Poynting vector of the dominant near field terms, we find

\[
S = \frac{1}{2} E \times H^* = \frac{1}{2} \left[ (E_r a_r + E_\theta a_\theta) \times H^*_\phi a_\phi \right] = \frac{1}{2} \left[ -E_r H^*_\phi a_\theta + E_\theta H^*_\phi a_r \right]
\]

\[
= j\eta \left( \frac{I_o \Delta l}{4\pi} \right)^2 \left[ \frac{\sin\theta \cos\theta}{r^5} a_\theta - \frac{\sin^2\theta}{2r^5} a_r \right]
\]

The Poynting vector (complex vector power density) for the infinitesimal dipole near field is purely imaginary. An imaginary Poynting vector corresponds to standing waves or stored energy (reactive power).
The vector form of the near electric field is the same as that for an electrostatic dipole (charges $+q$ and $-q$ separated by a distance $\Delta l$).

If we replace the term $(I_o \eta/k)$ by in the near electric field terms by its charge equivalent expression, we find

\[ I_o = \frac{\partial q}{\partial t} = j \omega q \quad I_o \frac{\eta}{k} = j \omega q \sqrt{\mu \frac{1}{\epsilon \omega \sqrt{\mu \epsilon}}} = j \frac{q}{\epsilon} \]

The electric field expression above is identical to that of the electrostatic dipole except for the complex exponential term (the infinitesimal dipole electric field oscillates). This result is related to the assumption of a uniform current over the length of the infinitesimal dipole. The only way for the current to be uniform, even at the ends of the wire, is for charge to build up and decay at the ends of the dipole as the current oscillates.

The near magnetic field of the infinitesimal dipole can be shown to be mathematically equivalent to that of a short DC current segment multiplied by the same complex exponential term.
Radiating near field \([ kr \approx 1 \text{ or } r \approx \lambda/2\pi ] \)

The dominant terms for the radiating near field of the infinitesimal dipole are the terms which are constant with respect to \( kr \) for \( E_\theta \) and \( H_\phi \) and the term proportional to \((kr)^{-1}\) for \( E_r \).

\[
E_r \approx \eta \frac{I_o \Delta l e^{-jkr}}{2\pi r^2} \cos \theta
\]

\[
E_\theta \approx j\eta \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \theta
\]

\[
H_\phi \approx j \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \theta
\]

Note that \( E_\theta \) and \( H_\phi \) are now in phase which yields a Poynting vector for these two components which is purely real (radiation). The direction of this component of the Poynting vector is outward radially denoting the outward radiating real power.

Far field \([ kr \gg 1 \text{ or } r \gg \lambda/2\pi ] \)

The dominant terms for the far field of the infinitesimal dipole are the terms which are constant with respect to \( kr \).

\[
E_\theta \approx j \eta \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \theta
\]

\[
H_\phi \approx j \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \theta
\]
Note that the far field components of $E$ and $H$ are the same two components which produced the radially-directed real-valued Poynting vector (radiated power) for the radiating near field. Also note that there is no radial component of $E$ or $H$ so that the propagating wave is a transverse electromagnetic (TEM) wave. For very large values of $r$, this TEM wave approaches a plane wave. The ratio of the far electric field to the far magnetic field for the infinitesimal dipole yields the intrinsic impedance of the medium.

\[
\frac{E_\theta}{H_\phi} \approx \eta
\]
Far Field of an Arbitrarily Oriented Infinitesimal Dipole

Given the equations for the far field of an infinitesimal dipole oriented along the \( z \)-axis, we may generalize these equations for an infinitesimal dipole antenna oriented in any direction. The far fields of infinitesimal dipole oriented along the \( z \)-axis are

\[
E_\theta \approx j \eta \frac{kI_o \Delta l e^{-jk r}}{4 \pi r} \sin \theta \\
H_\phi \approx j \frac{kI_o \Delta l e^{-jk r}}{4 \pi r} \sin \theta
\]

If we rotate the antenna by some arbitrary angle \( \alpha \) and define the new direction of the current flow by the unit vector \( a_\alpha \), the resulting far fields are simply a rotated version of the original equations above. In the rotated coordinate system, we must define new angles \((\alpha, \beta)\) that correspond to the spherical coordinate angles \((\theta, \phi)\) in the original coordinate system. The angle \( \beta \) is shown below referenced to the \( x \)-axis (as \( \phi \) is defined) but can be referenced to any convenient axis that could represent a rotation in the \( \phi \)-direction.
Note that the infinitesimal far fields in the original coordinate system depend on the spherical coordinates \( r \) and \( \theta \). The value of \( r \) is identical in the two coordinate systems since it represents the distance from the coordinate origin. However, we must determine the transformation from \( \theta \) to \( \alpha \). The transformations of the far fields in the original coordinate system to those in the rotated coordinate system can be written as

\[
E_\theta \approx j \eta \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \theta \quad \Rightarrow \quad E_\alpha \approx j \eta \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \alpha
\]

\[
H_\phi \approx j \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \theta \quad \Rightarrow \quad H_\beta \approx j \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sin \alpha
\]

Specifically, we need the definition of \( \sin \alpha \). According to the trigonometric identity

\[
\sin^2 \alpha + \cos^2 \alpha = 1
\]

we may write

\[
\sin \alpha = \sqrt{1 - \cos^2 \alpha}
\]

Based on the definition of the dot product, the \( \cos \alpha \) term may be written as

\[
\cos \alpha = a_r \cdot a_\alpha
\]

so that

\[
\sin \alpha = \sqrt{1 - (a_r \cdot a_\alpha)^2}
\]

Inserting our result for the \( \sin \alpha \) term yields

\[
E_\alpha \approx j \eta \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sqrt{1 - (a_r \cdot a_\alpha)^2}
\]

\[
H_\beta \approx j \frac{kI_o \Delta l e^{-jkr}}{4\pi r} \sqrt{1 - (a_r \cdot a_\alpha)^2}
\]
Example

Determine the far fields of an infinitesimal dipole oriented along the $y$-axis.

\[ a_r \cdot a_\alpha = a_r \cdot a_y = \sin \theta \sin \phi \]

\[ E_\alpha \approx j \eta \frac{k I_o \Delta l e^{-jkr}}{4\pi r} \sqrt{1 - (\sin \theta \sin \phi)^2} \]

\[ H_\beta \approx j \frac{k I_o \Delta l e^{-jkr}}{4\pi r} \sqrt{1 - (\sin \theta \sin \phi)^2} \]
Poynting’s Theorem (Conservation of Power)

Poynting’s theorem defines the basic principle of conservation of power which may be applied to radiating antennas. The derivation of the time-harmonic form of Poynting’s vector begins with the following vector identity

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

If we insert the Poynting vector ($S = E \times H^*$) in the left hand side of the above identity, we find

$$\nabla \cdot (E \times H^*) = H^* \cdot (\nabla \times E) - E \cdot (\nabla \times H^*)$$

From Maxwell’s equations, the curl of $E$ and $H$ are

$$\nabla \times H = j \omega \epsilon E + J$$
$$\nabla \times E = -j \omega \mu H$$

such that

$$\nabla \cdot (E \times H^*) = H^* \cdot (-j \omega \mu H) - E \cdot (-j \omega \epsilon E^* + J^*)$$
$$= -j \omega \mu |H|^2 + j \omega \epsilon |E|^2 - E \cdot J^*$$

Integrating both sides of this equation over any volume $V$ and applying the divergence theorem to the left hand side gives

$$\int \int \int_{V} [\nabla \cdot (E \times H^*)] dv = \int_{S} (E \times H^*) \cdot ds$$
$$\quad = -j \omega \left[ \int \int \int_{V} \mu |H|^2 dv - \int \int \int_{V} \epsilon |E|^2 dv \right] - \int \int \int_{V} (E \cdot J^*) dv$$

The current density in the equation above consists of two components: the impressed (source) current ($J_i$) and the conduction current ($J_c$).

$$J = J_c + J_i = \sigma E + J_i$$
Inserting the current expression and dividing both sides of the equation by 2 yields Poynting’s theorem.

\[ \frac{1}{2} \iiint_S (E \times H^*) \cdot ds = -j \omega \left[ \iiint_V \frac{1}{2} \mu |H|^2 \, dv - \iiint_V \frac{1}{2} \varepsilon |E|^2 \, dv \right] \]

\[ - \frac{1}{2} \iiint_V \sigma |E|^2 \, dv - \frac{1}{2} \iiint_V (E \cdot J_i^*) \, dv \]

The individual terms in the above equation may be identified as

\[ P = \frac{1}{2} \iiint_S (E \times H^*) \cdot ds \quad \text{(complex power flowing out of } V) \]

\[ P_{\text{rad}} = \frac{1}{2} \text{Re} \iiint_S (E \times H^*) \cdot ds \quad \text{(real power flowing out of } V) \]

\[ W_m = \frac{1}{2} \iiint_V \left( \frac{1}{2} \mu |H|^2 \right) \, dv \quad \text{(time average stored magnetic energy within } V) \]

\[ W_e = \frac{1}{2} \iiint_V \left( \frac{1}{2} \varepsilon |E|^2 \right) \, dv \quad \text{(time average stored electric energy within } V) \]

\[ P_L = \frac{1}{2} \iiint_V (\sigma |E|^2) \, dv \quad \text{(time average dissipated power energy within } V) \]

\[ P_S = -\frac{1}{2} \iiint_V (E \cdot J_i^*) \, dv \quad \text{(power delivered by sources within } V) \]

Poynting’s theorem may then be written as

\[ P_S = P + P_L + j \omega (W_m - W_e) \]
Total Power and Radiation Resistance

To determine the total complex power (radiated plus reactive) produced by the infinitesimal dipole, we integrate the Poynting vector over a spherical surface enclosing the antenna. We must use the complete field expressions to determine both the radiated and reactive power. The time-average complex Poynting vector is

\[
S = \frac{1}{2} \mathbf{E} \times \mathbf{H}^* = \frac{1}{2} \left[ (E_r a_r + E_\theta a_\theta) \times H_\phi^* a_\phi \right] = \frac{1}{2} \left[ -E_r H_\phi^* a_\theta + E_\theta H_\phi^* a_r \right]
\]

The total complex power passing through the spherical surface of radius \( r \) is found by integrating the normal component of the Poynting vector over the surface.

\[
P = \int \int_S S \cdot ds = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \left[ E_\theta H_\phi^* a_r - E_r H_\phi^* a_\theta \right] \cdot a_r r^2 \sin \theta d\theta d\phi
\]

\[
= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} E_\theta H_\phi^* r^2 \sin \theta d\theta d\phi = P_{rad} + j2 \omega (W_m - W_e)
\]
The terms $W_e'$ and $W_m'$ represent the radial electric and magnetic energy flow through the spherical surface $S$.

\[ E_\theta H_\phi^* = \left\{ \eta \frac{kI_o \Delta l}{4\pi r} \sin\theta \left[ j + \frac{1}{kr} - \frac{j}{(kr)^2} \right] e^{-jkr} \right\} \times \left\{ \frac{kI_o^* \Delta l}{4\pi r} \sin\theta \left[ -j + \frac{1}{kr} \right] e^{jkr} \right\} \]

\[ = \eta \left( \frac{k |I_o| \Delta l}{4\pi r} \sin\theta \right)^2 \left[ 1 + \frac{j}{kr} - \frac{j}{kr} + \frac{1}{(kr)^2} - \frac{1}{(kr)^2} - \frac{j}{(kr)^3} \right] \]

\[ = \eta \left( \frac{k |I_o| \Delta l}{4\pi r} \sin\theta \right)^2 \left[ 1 - \frac{j}{(kr)^3} \right] \]

The total power through the sphere is

\[ P = \frac{\eta}{2} \left( \frac{k |I_o| \Delta l}{4\pi} \right)^2 \left[ 1 - \frac{j}{(kr)^3} \right] \int_0^{2\pi} \int_0^\pi \sin^3\theta \, d\theta \, d\phi \]

\[ = \frac{\eta}{2} \left( \frac{k |I_o| \Delta l}{4\pi} \right)^2 \left[ 1 - \frac{j}{(kr)^3} \right] (2\pi) \left[ -\frac{1}{3} \cos\theta (\sin^2\theta + 2) \right]_0^{\pi} \]

\[ = \frac{\eta \pi}{3} \left( \frac{k |I_o| \Delta l}{4\pi} \right)^2 \left[ 1 - \frac{j}{(kr)^3} \right] \]

\[ = \frac{\eta \pi}{3} \left( \frac{|I_o| \Delta l}{\lambda} \right)^2 \left[ 1 - \frac{j}{(kr)^3} \right] \]

\[ k = \frac{2\pi}{\lambda} \]
The real and imaginary parts of the complex power are

\[ P_{rad} = \eta \frac{\pi}{3} \left( \frac{|I_o| \Delta l}{\lambda} \right)^2 = \eta \frac{\pi}{3} \left( \frac{k|I_o| \Delta l}{2\pi} \right)^2 \]

\[ 2\omega (W_m' - W_e') = -\eta \frac{\pi}{3} \left( \frac{|I_o| \Delta l}{\lambda} \right)^2 \frac{1}{(kr)^3} \]

The radiation resistance for the infinitesimal dipole is found according to

\[ P_{rad} = \eta \frac{\pi}{3} \left( \frac{|I_o| \Delta l}{\lambda} \right)^2 = \frac{1}{2} |I_o|^2 R_r \]

\[ R_r = \eta \frac{2\pi}{3} \left( \frac{\Delta l}{\lambda} \right)^2 = 80\pi^2 \left( \frac{\Delta l}{\lambda} \right)^2 \]

Infinitesimal dipole radiation resistance
Infinitesimal Dipole Radiation Intensity and Directivity

The radiation intensity of the infinitesimal dipole may be found by using the previously determined total fields.

\[ P_{\text{rad}} = \frac{1}{2} \text{Re} \iint_S (E \times H^*) \cdot ds \]

\[ = \frac{1}{2} \text{Re} \int_0^{2\pi} \int_0^{2\pi} E_\theta H_\phi^* r^2 \sin \theta \ d\theta \ d\phi \]

\[ = \int_0^{2\pi} \int_0^{2\pi} U(\theta, \phi) \sin \theta \ d\theta \ d\phi \]

\[ U(\theta, \phi) = \frac{r^2}{2} \text{Re} \left[ E_\theta H_\phi^* \right] \]

\[ = \frac{r^2}{2} \eta \left[ k |I_o| \Delta l \frac{\sin \theta}{4\pi r} \right]^2 \]

\[ = \frac{\eta}{2} \left[ k |I_o| \Delta l \frac{\sin \theta}{4\pi} \right]^2 \]

\[ D(\theta, \phi) = 4\pi \frac{U(\theta, \phi)}{P_{\text{rad}}} = 4\pi \frac{\eta}{2} \left[ k |I_o| \Delta l \frac{\sin \theta}{4\pi} \right]^2 \]

\[ \frac{\eta}{3} \left[ k |I_o| \Delta l \right] = 1.5 \sin^2 \theta \]

\[ D(\theta, \phi) = 1.5 \sin^2 \theta \]

Infinitesimal dipole directivity function

\[ D_o = [D(\theta, \phi)]_{\text{max}} = 1.5 \]

Infinitesimal dipole Maximum directivity
Infinitesimal Dipole Effective Aperture and Solid Beam Angle

The effective aperture of the infinitesimal dipole is found from the maximum directivity:

\[ A_e = \frac{\lambda^2}{4\pi} D_o = \frac{\lambda^2}{4\pi} \frac{3}{2} \]

\[ A_e = \frac{3 \lambda^2}{8\pi} \text{ Infinitesimal dipole effective aperture} \]

The beam solid angle for the infinitesimal dipole can be found from the maximum directivity,

\[ \Omega_A = \frac{4\pi}{D_o} = \frac{4\pi}{3} = \frac{8\pi}{3} \]

or can be determined directly from the radiation intensity function.

\[ \Omega_A = \int_0^{2\pi} \int_0^{\pi} \frac{U(\theta, \phi)}{[U(\theta, \phi)]_{\text{max}}} \sin\theta \, d\theta \, d\phi \]

\[ U_{\text{max}}(\theta, \phi) = \frac{\eta}{2} \left[ \frac{k |I_o| \Delta l}{4\pi} \right]^2 \]

\[ \Omega_A = \int_0^{2\pi} \int_0^{\pi} (\sin^2\theta) \, \sin\theta \, d\theta \, d\phi \]

\[ = 2\pi \int_0^\pi \sin^3\theta \, d\theta = 2\pi \frac{4}{3} \]

\[ \Omega_A = \frac{8\pi}{3} \text{ Infinitesimal dipole beam solid angle} \]
Short Dipole

\( (\lambda/50 \leq l \leq \lambda/10, \ a \ll \lambda) \)

\[
I(z') = I_o \left(1 - \frac{2|z'|}{l}\right) a_z \quad \left(-\frac{l}{2} \leq z' \leq \frac{l}{2}\right)
\]

\[
= \begin{cases} 
I_o \left(1 + \frac{2z'}{l}\right) a_z & \left(-\frac{l}{2} \leq z' \leq 0\right) \\
I_o \left(1 - \frac{2z'}{l}\right) a_z & \left(0 \leq z' \leq \frac{l}{2}\right)
\end{cases}
\]
Note that the magnetic vector potential of the short dipole (length = \(l\), peak current = \(I_o\)) is one half that of the equivalent infinitesimal dipole (length \(\Delta l = l\), current = \(I_o\)).

\[
A(r) = \frac{\mu}{4\pi} \int \int J(r') \frac{e^{-jk|r-r'|}}{|r-r'|} \, dv' \\
J(r') \rightarrow I(z') \\
dv' \rightarrow dl' = dz' \quad \left( -\frac{l}{2} \leq z' \leq \frac{l}{2}\right) \\
r' \approx 0 \quad \text{(electrically small)} \\
|r-r'| \approx |r| = r \\
A(r) \approx \frac{\mu}{4\pi} \left\{ \int_{-l/2}^{0} I_o \left( 1 + \frac{2z'}{l} \right) a_z \frac{e^{-jkr}}{r} dz' + \int_{0}^{l/2} I_o \left( 1 - \frac{2z'}{l} \right) a_z \frac{e^{-jkr}}{r} dz' \right\} \\
= \frac{\mu I_o}{4\pi r} e^{-jkr} a_z \left\{ \left[ \frac{z'}{l} + \frac{z^2}{l} \right]_{-l/2}^{0} + \left[ \frac{z'}{l} - \frac{z^2}{l} \right]_{0}^{l/2} \right\} \\
= \frac{\mu I_o}{4\pi r} e^{-jkr} a_z \left\{ \frac{l}{2} - \frac{l}{4} + \frac{l}{2} - \frac{l}{4} \right\} \\
= \frac{\mu I_o}{8\pi r} e^{-jkr} a_z \\

A_{\text{short dipole}} = \frac{1}{2} A_{\text{infinitesimal dipole}}
The average current on the short dipole is one half that of the equivalent infinitesimal dipole. Therefore, the fields produced by the short dipole are exactly one half those produced by the equivalent infinitesimal dipole.

\[
E_r = \eta \frac{kI_o l}{4\pi r} \cos\theta \left[ 1 - \frac{j}{kr} - \frac{j}{(kr)^2} \right] e^{-jkr}
\]

\[
E_\theta = \eta \frac{kI_o l}{8\pi r} \sin\theta \left[ j + \frac{1}{kr} - \frac{j}{(kr)^2} \right] e^{-jkr}
\]

\[
H_\phi = \frac{kI_o l}{8\pi r} \sin\theta \left[ j + \frac{1}{kr} \right] e^{-jkr}
\]

\[
E_r = -j\eta \frac{I_o l e^{-jkr}}{4\pi kr^3} \cos\theta
\]

\[
E_\theta = -j\eta \frac{I_o l e^{-jkr}}{8\pi kr^3} \sin\theta
\]

\[
H_\phi = \frac{I_o l e^{-jkr}}{8\pi r^2} \sin\theta
\]

\[
E_r \approx \eta \frac{I_o l e^{-jkr}}{4\pi r^2} \cos\theta
\]

\[
E_\theta \approx j\eta \frac{kI_o l e^{-jkr}}{8\pi r} \sin\theta
\]

\[
H_\phi \approx j \frac{kI_o l e^{-jkr}}{8\pi r} \sin\theta
\]
Since the fields produced by the short dipole are one half those of the equivalent infinitesimal dipole, the real power radiated by the short dipole is one fourth that of the infinitesimal dipole. Thus, $P_{\text{rad}}$ for the short dipole is

$$P_{\text{rad}} = \eta \frac{\pi}{12} \left( \frac{|I_o| \Delta l}{\lambda} \right)^2 = \frac{1}{2} |I_o|^2 R_r$$

and the associated radiation resistance is

$$R_r = \eta \frac{\pi}{6} \left( \frac{l}{\lambda} \right)^2 = 20 \pi^2 \left( \frac{l}{\lambda} \right)^2$$

The directivity function, the maximum directivity, effective area and beam solid angle of the short dipole are all identical to the corresponding value for the infinitesimal dipole.
If we assume that the dipole antenna is driven at its center, we may assume that the current distribution is symmetrical along the antenna.

\[
I(z') = I_o \sin \left( k \left( \frac{l}{2} - |z'| \right) \right) a_z \quad \left( -\frac{l}{2} \leq z' \leq \frac{l}{2} \right)
\]

\[
= \begin{cases} 
I_o \sin \left[ k \left( \frac{l}{2} + z' \right) \right] a_z & \left( -\frac{l}{2} \leq z' \leq 0 \right) \\
I_o \sin \left[ k \left( \frac{l}{2} - z' \right) \right] a_z & \left( 0 \leq z' \leq \frac{l}{2} \right)
\end{cases}
\]

We use the previously defined approximations for the far field magnetic vector potential to determine the far fields of the center-fed dipole.

\[
A(r) \approx \mu \frac{e^{-jkr}}{4 \pi r} \int \int \int \mathbf{J}(r') e^{jkr' \cos \alpha} d\nu'
\]
For the center-fed dipole lying along the $z$-axis, $x' = y' = 0$, so that

$$ r' \cos \alpha = z' \cos \theta $$

$$ A(r) \approx \frac{\mu}{4 \pi r} \frac{e^{-jkr}}{r} \int_{-\frac{1}{2}}^{l+\frac{1}{2}} I(z') e^{jkz' \cos \theta} \, dz' $$

$$ = \frac{\mu}{4 \pi r} I_o a_z \left\{ \int_{-\frac{1}{2}}^{0} \sin \left[ k \left( \frac{l}{2} + z' \right) \right] e^{jkz' \cos \theta} \, dz' + \int_{0}^{l+\frac{1}{2}} \sin \left[ k \left( \frac{l}{2} - z' \right) \right] e^{jkz' \cos \theta} \, dz' \right\} $$
\[ \int \sin(a + bx) e^{cx} \, dx = \frac{e^{cx}}{b^2 + c^2} \left[ c \sin(a + bx) - b \cos(a + bx) \right] \]

\[ A(r) \approx \mu \frac{e^{-jkr}}{4 \pi r} I_o a_z \left[ \frac{e^{j k z' \cos \theta}}{k^2 - k^2 \cos^2 \theta} \right] \]

\[ \times \left\{ (j k \cos \theta) \sin \left[ k \left( \frac{l}{2} + z' \right) \right] - k \cos \left[ k \left( \frac{l}{2} + z' \right) \right] \right\}^{0} \]

\[ \times \left\{ (j k \cos \theta) \sin \left[ k \left( \frac{l}{2} - z' \right) \right] + k \cos \left[ k \left( \frac{l}{2} - z' \right) \right] \right\}^{1/2} \]

\[ A(r) \approx \mu \frac{e^{-jkr}}{4 \pi r} I_o a_z \left[ \frac{1}{k^2 \sin^2 \theta} \right] \]

\[ \times \left\{ (j k \cos \theta) \sin \left[ k \frac{l}{2} \right] - k \cos \left[ k \frac{l}{2} \right] + k e^{-j k \frac{l}{2} \cos \theta} \right\} \]

\[ + k e^{-j k \frac{l}{2} \cos \theta} - (j k \cos \theta) \sin \left[ k \frac{l}{2} \right] - k \cos \left[ k \frac{l}{2} \right] \right\} \]

\[ A(r) \approx \mu \frac{e^{-jkr}}{4 \pi r} I_o a_z \frac{1}{k^2 \sin^2 \theta} \left\{ 2 k \cos \left( k \frac{l}{2} \cos \theta \right) - 2 k \cos \left( k \frac{l}{2} \right) \right\} \]

\[ A(r) \approx \mu \frac{e^{-jkr}}{2 \pi r} I_o a_z \left[ \frac{\cos \left( k \frac{l}{2} \cos \theta \right) - \cos \left( k \frac{l}{2} \right)}{k \sin^2 \theta} \right] \]
Transforming the $z$-directed vector potential to spherical coordinates gives

$$A_z \mathbf{a}_z = A_z \cos \theta \mathbf{a}_r - A_z \sin \theta \mathbf{a}_\theta$$

$$A(r) \approx \frac{\mu}{2\pi r} e^{-jkr} I_o \left( \cos \theta \mathbf{a}_r - \sin \theta \mathbf{a}_\theta \right) \left[ \cos \left( \frac{kl \cos \theta}{2} \right) - \cos \left( \frac{kl}{2} \right) \right]$$

(Center-fed dipole far field magnetic vector potential)

The far fields of the center-fed dipole in terms of the magnetic vector potential are

$$E_\theta \approx -j \omega A_\theta$$

$$E_\theta \approx \frac{j \omega \mu}{k} e^{-jkr} I_o \left[ \cos \left( \frac{kl \cos \theta}{2} \right) - \cos \left( \frac{kl}{2} \right) \right]$$

(Center-fed dipole far field electric field)

$$H_\phi \approx -j \frac{\omega}{\eta} A_\theta = \frac{E_\theta}{\eta}$$

$$H_\phi \approx j \frac{e^{-jkr} I_o}{2\pi r} \left[ \cos \left( \frac{kl \cos \theta}{2} \right) - \cos \left( \frac{kl}{2} \right) \right]$$

(Center-fed dipole far field magnetic field)
The time-average complex Poynting vector in the far field of the center-fed dipole is

\[ S = \frac{1}{2} E \times H^* = \frac{1}{2} \left[ E_\theta a_\theta \times H_\phi^* a_\phi \right] = \frac{1}{2} \left[ E_\theta \frac{E_\theta^*}{\eta} a_r \right] = \frac{|E_\theta|^2}{2 \eta} a_r \]

\[ = \frac{\eta |I_o|^2}{8 \pi^2 r^2} \left[ \frac{\cos \left( \frac{kl}{2} \cos \theta \right) - \cos \left( \frac{kl}{2} \right)}{\sin \theta} \right]^2 a_r \]

The radiation intensity function for the center-fed dipole is given by

\[ U(\theta,\phi) = r^2 S \]

\[ U(\theta,\phi) = \frac{\eta |I_o|^2}{8 \pi^2} \left[ \frac{\cos \left( \frac{kl}{2} \cos \theta \right) - \cos \left( \frac{kl}{2} \right)}{\sin \theta} \right]^2 \]

(Center-fed dipole radiation intensity function)

We may plot the normalized radiation intensity function \([U(\theta) = B_o F(\theta)]\) to determine the effect of the antenna length on its radiation pattern.

\[ F(\theta) = \left[ \frac{\cos \left( \frac{kl}{2} \cos \theta \right) - \cos \left( \frac{kl}{2} \right)}{\sin \theta} \right]^2 \]
In general, we see that the directivity of the antenna increases as the length goes from a short dipole (a fraction of a wavelength) to a full wavelength. As the length increases above a wavelength, more lobes are introduced into the radiation pattern.
\[ l = \lambda/10 \]

\[ I(z) / I_0 \]

\[ l = \lambda/2 \]

\[ I(z) / I_0 \]

\[ l = \lambda \]

\[ I(z) / I_0 \]

\[ l = 3\lambda/2 \]

\[ I(z) / I_0 \]
The total real power radiated by the center-fed dipole is

\[
P_{\text{rad}} = \frac{2\pi}{4\pi} \int_0^{2\pi} \int_0^\infty U(\theta, \phi) \sin \theta \, d\theta \, d\phi
\]

The \(\theta\)-dependent integral in the radiated power expression cannot be integrated analytically. However, the integral may be manipulated, using several transformations of variables, into a form containing some commonly encountered special functions (integrals) known as the sine integral and cosine integral.

\[
S_i(x) = \int_0^x \frac{\sin x}{x} \, dx \quad \text{(sine integral)}
\]

\[
C_i(x) = -\int_x^\infty \frac{\cos x}{x} \, dx \quad \text{(cosine integral)}
\]

\[
C_{\text{in}}(x) = \int_0^x \left( \frac{1 - \cos x}{x} \right) \, dx 
= C + \ln(x) - C_i(x)
\]

\[
C = 0.5772 \quad \text{(Euler's constant)}
\]

The radiated power of the center-fed dipole becomes

\[
P_{\text{rad}} = \frac{\eta |I_o|^2}{4\pi} \left\{ C + \ln(kl) - C_i(kl) + \frac{1}{2} \sin(kl) \left[ S_i(2kl) - 2S_i(kl) \right] 
+ \frac{1}{2} \cos(kl) \left[ C + \ln(kl/2) + C_i(2kl) - 2C_i(kl) \right] \right\}
\]
The radiated power is related to the radiation resistance of the antenna by

\[ P_{\text{rad}} = \frac{1}{2} |I_o|^2 R_r \]

which gives

\[
R_r = \frac{\eta}{2\pi} \left\{ C + \ln(kl) - C_i(kl) + \frac{1}{2} \sin(kl) \left[ S_i(2kl) - 2S_i(kl) \right] \right. \\
\left. + \frac{1}{2} \cos(kl) \left[ C + \ln(kl/2) + C_i(2kl) - 2C_i(kl) \right] \right\}
\]

(Center-fed dipole radiation resistance)

The directivity function of the center-fed dipole is given by

\[
D(\theta, \phi) = 4\pi \frac{U(\theta, \phi)}{P_{\text{rad}}}
\]

\[
U(\theta, \phi) = \frac{\eta |I_o|^2}{8\pi^2} \left[ \cos \left( \frac{kl}{2} \cos \theta \right) - \cos \left( \frac{kl}{2} \right) \right]^2 = \frac{\eta |I_o|^2}{8\pi^2} F(\theta, \phi)
\]

\[
P_{\text{rad}} = \frac{\eta |I_o|^2}{4\pi} f(kl)
\]

\[
f(kl) = C + \ln(kl) - C_i(kl) + \frac{1}{2} \sin(kl) \left[ S_i(2kl) - 2S_i(kl) \right] \\
+ \frac{1}{2} \cos(kl) \left[ C + \ln(kl/2) + C_i(2kl) - 2C_i(kl) \right]
\]
Center-fed dipole directivity function

\[
D(\theta, \phi) = 4\pi \frac{1}{2\pi} \frac{F(\theta, \phi)}{f(kl)} = 2 \frac{F(\theta, \phi)}{f(kl)}
\]

The maximum directivity is

\[
D_o = 2 \frac{\left[F(\theta, \phi)\right]_{\text{max}}}{f(kl)}
\]

The effective aperture is

\[
A_e = \frac{\lambda^2}{4\pi} D_o
\]

\[
A_e = \frac{\lambda^2}{2\pi} \frac{\left[F(\theta, \phi)\right]_{\text{max}}}{f(kl)}
\]

Center-fed dipole effective aperture

\[
\Omega_A = \frac{4\pi}{D_o} = \frac{4\pi}{2 \frac{\left[F(\theta, \phi)\right]_{\text{max}}}{f(kl)}}
\]

\[
\Omega_A = 2\pi \frac{f(kl)}{\left[F(\theta, \phi)\right]_{\text{max}}}
\]

Center-fed dipole Solid beam angle
**Half-Wave Dipole**

The diagram illustrates a center-fed half-wave dipole with far fields.

Mathematical expressions for the far fields are given as:

\[ E_\theta \approx j \eta \frac{e^{-jkr}}{2\pi r} I_o \frac{\cos\left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \]

\[ H_\phi \approx j \frac{e^{-jkr}}{2\pi r} I_o \frac{\cos\left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \]

Center-fed half-wave dipole far fields

The radiation intensity function is:

\[ U(\theta, \phi) = \frac{\eta |I_o|^2 \cos^2\left(\frac{\pi}{2} \cos \theta \right)}{8\pi^2 \sin^2 \theta} \]

Center-fed half-wave dipole radiation intensity function
\[ P_{rad} = \frac{\eta |I_o|^2}{8 \pi} \left\{ C + 2 \ln(\pi) - \ln\left(\frac{\pi}{2}\right) - C_i(2\pi) \right\} \]

\[ = \frac{\eta |I_o|^2}{8 \pi} \left\{ \frac{C}{2} + \ln(\pi) - \frac{1}{2} \ln\left(\frac{\pi}{2}\right) - \frac{1}{2} C_i(2\pi) \right\} \]

\[ = \frac{\eta |I_o|^2}{8 \pi} \left\{ C + 2 \ln(\pi) - \ln\left(\frac{\pi}{2}\right) - C_i(2\pi) \right\} \]

\[ 2 \ln(\pi) - \ln\left(\frac{\pi}{2}\right) = \ln(\pi^2) - \ln\left(\frac{\pi}{2}\right) = \ln\left(\frac{\pi^2}{\pi/2}\right) = \ln(2\pi) \]

\[ P_{rad} = \frac{\eta |I_o|^2}{8 \pi} \left\{ C + \ln(2\pi) - C_i(2\pi) \right\} \]

\[ = \frac{\eta |I_o|^2}{8 \pi} C_{in}(2\pi) \]

\[ = \frac{\eta |I_o|^2}{8 \pi} (2.438) \quad \left[ f(\pi) = \frac{2.348}{2} = 1.219 \right] \]

\[ = 36.57 |I_o|^2 \quad \text{(half-wave dipole in air, } \eta = 120\pi) \]

\[ = \frac{1}{2} |I_o|^2 R_r \]

\[ R_{rad} = 2(36.57) = 73.14 \Omega \]

Center-fed half-wave dipole radiation resistance (in air)
\[ D(\theta, \phi) = 2 \frac{F(\theta, \phi)}{f(\pi)} = 2 \frac{\cos^2\left(\frac{\pi}{2} \cos\theta\right)}{(1.219) \sin^2\theta} = 1.641 \frac{\cos^2\left(\frac{\pi}{2} \cos\theta\right)}{\sin^2\theta} \]

\[ D(\theta, \phi) = 1.641 \ \text{F(\theta, \phi) = 1.641} \frac{\cos^2\left(\frac{\pi}{2} \cos\theta\right)}{\sin^2\theta} \]

Center-fed half-wave dipole directivity function

\[ D_o = 1.641 \ [F(\theta, \phi)]_{\max} \]

\[ [F(\theta, \phi)]_{\max} = F(\theta, \phi) \big|_{\theta = \frac{\pi}{2}} = \frac{\cos^2\left[\left(\frac{\pi}{2}\right)(0)\right]}{(1)} = 1 \]

\[ D_o = 1.641 \ (2.15 \ \text{dB}) \] Center-fed half-wave dipole maximum directivity

\[ A_e = \frac{\lambda^2}{2\pi} \frac{[F(\theta, \phi)]_{\max}}{f(kl)} = \frac{\lambda^2}{2\pi} \frac{1}{1.219} \]

\[ A_e = 0.131 \lambda^2 \] Center-fed half-wave dipole effective aperture
Dipole Input Impedance

The input impedance of the dipole is defined as the ratio of voltage to current at the antenna feed point.

\[ Z_{in} = \frac{V_{in}}{I_{in}} = R_{in} + jX_{in} \]

The real and reactive time-average power delivered to the terminals of the antenna may be written as

\[
Re\{P_{in}\} = \frac{1}{2} Re\{V_{in}I_{in}^*\} = \frac{1}{2} Re\{ |I_{in}|^2 Z_{in} \} = \frac{1}{2} |I_{in}|^2 R_{in}
\]

\[
Im\{P_{in}\} = \frac{1}{2} Im\{V_{in}I_{in}^*\} = \frac{1}{2} Im\{ |I_{in}|^2 Z_{in} \} = \frac{1}{2} |I_{in}|^2 X_{in}
\]

If we assume that the antenna is lossless \((R_L = 0)\), then the real power delivered to the input terminals equals that radiated by the antenna. Thus,

\[
\frac{1}{2} |I_{in}|^2 R_{in} = \frac{1}{2} |I_o|^2 R_r
\]
and the antenna input resistance is related to the antenna radiation resistance by
\[ R_{in} = \frac{|I_o|^2}{|I_{in}|^2} R_r \]

In a similar fashion, we may equate the reactive power delivered to the antenna input terminals to that stored in the near field of the antenna.
\[ \frac{1}{2} |I_{in}|^2 X_{in} = \frac{1}{2} |I_o|^2 X_A \]
or
\[ X_{in} = \frac{|I_o|^2}{|I_{in}|^2} X_A \]

The general dipole current is defined by
\[ I(z') = I_o \sin \left[ k \left( \frac{l}{2} - |z'| \right) \right] a_z \quad \left( -\frac{l}{2} \leq z' \leq \frac{l}{2} \right) \]
The current \( I_{in} \) is the current at the feed point of the dipole \((z' = 0)\) so that
\[ I_{in} = I(0) = I_o \sin \left( \frac{kl}{2} \right) \]
The input resistance and reactance of the antenna are then related to the equivalent circuit values of radiation resistance and the antenna reactance by
\[ R_{in} = \frac{R_r}{\sin^2 \left( \frac{kl}{2} \right)} \quad X_{in} = \frac{X_A}{\sin^2 \left( \frac{kl}{2} \right)} \]
The dipole reactance may be determined in closed form using a technique known as the induced EMF method (Chapter 8) but requires that the radius of the wire \((a)\) be included. The resulting dipole reactance is

\[
X_A = \frac{\eta}{4\pi} \left\{ 2S_i(kl) + \cos(kl) \left[ 2S_i(kl) - S'_i(2kl) \right] \\
- \sin(kl) \left[ 2C_i(kl) - C'_i(2kl) - C'_i \left( \frac{2ka^2}{l} \right) \right] \right\}
\]

(Center-fed dipole reactance)

The input resistance and reactance are plotted in Figure 8.16 (p.411) for a dipole of radius \(a = 10^{-5}\lambda\). If the dipole is 0.5\(\lambda\) in length, the input impedance is found to be approximately \((73 + j42.5)\ \Omega\). The first dipole resonance \((X_{in} = 0)\) occurs when the dipole length is slightly less than one-half wavelength. The exact resonant length depends on the wire radius, but for wires that are electrically very thin, the resonant length of the dipole is approximately 0.48\(\lambda\). As the wire radius increases, the resonant length decreases slightly [see Figure 8.17 (p.412)].
Antenna and Scatterers

All of the antennas considered thus far have been assumed to be radiating in a homogeneous medium of infinite extent. When an antenna radiates in the presence of a conductor (inhomogeneous medium), currents are induced on the conductor which re-radiate (scatter) additional fields. The total fields produced by an antenna in the presence of a scatterer are the superposition of the original radiated fields (incident fields, \([E^{inc}, H^{inc}]\) those produced by the antenna in the absence of the scatterer) plus the fields produced by the currents induced on the scatterer (scattered fields, \([E^{scat}, H^{scat}]\)).

To evaluate the total fields, we must first determine the scattered fields which depend on the currents flowing on the scatterer. The determination of the scatterer currents typically requires a numerical scheme (integral equation in terms of the scatterer currents or a differential equation in the form of a boundary value problem). However, for simple scatterer shapes, we may use image theory to simplify the problem.
Image Theory

Given an antenna radiating over a perfect conducting ground plane, [perfect electric conductor (PEC), perfect magnetic conductor (PMC)] we may use image theory to formulate the total fields without ever having to determine the surface currents induced on the ground plane. Image theory is based on the electric or magnetic field boundary condition on the surface of the perfect conductor (the tangential electric field is zero on the surface of a PEC, the tangential magnetic field is zero on the surface of a PMC). Using image theory, the ground plane can be replaced by the equivalent image current located an equal distance below the ground plane. The original current and its image radiate in a homogeneous medium of infinite extent and we may use the corresponding homogeneous medium equations.

Example (vertical electric dipole)
Currents over a PEC

Vertical electric current  
\[\uparrow\]

Horizontal electric current  
\[\rightarrow\]

Vertical magnetic current  
\[\uparrow\]

Horizontal magnetic current  
\[\rightarrow\]

PEC \( (\sigma = \infty) \)

---

Image Currents

Currents over a PMC

Vertical electric current  
\[\uparrow\]

Horizontal electric current  
\[\rightarrow\]

Vertical magnetic current  
\[\uparrow\]

Horizontal magnetic current  
\[\rightarrow\]

PMC \( (\sigma_m = \infty) \)

---

Image Currents
**Vertical Infinitesimal Dipole Over Ground**

Give a vertical infinitesimal electric dipole (z-directed) located a distance $h$ over a PEC ground plane, we may use image theory to determine the overall radiated fields.

The individual contributions to the electric field by the original dipole and its image are

$$E_{\theta_1} \approx j \eta \frac{k I_0 \Delta l e^{-jkr_1}}{4 \pi r_1} \sin \theta_1$$

$$E_{\theta_2} \approx j \eta \frac{k I_0 \Delta l e^{-jkr_2}}{4 \pi r_2} \sin \theta_2$$

In the far field, the lines defining $r, r_1$ and $r_2$ become almost parallel so that

$$\theta_1 \approx \theta_2 \approx \theta$$

$$r_1 \approx r - h \cos \theta$$

$$r_2 \approx r + h \cos \theta$$
The previous expressions for \( r_1 \) and \( r_2 \) are necessary for the phase terms in the dipole electric field expressions. But, for amplitude terms, we may assume that \( r_1 \approx r_2 \approx r \). The total field becomes

\[
E_\theta = E_{\theta 1} + E_{\theta 2} \\
\approx j \eta \frac{k l_o \Delta l}{4 \pi r} \left[ e^{-jk(r-h)} + e^{-jk(r+h)} \right] \sin \theta \\
\approx j \eta \frac{k l_o \Delta l e^{-jkr}}{4 \pi r} \left[ e^{jkh \cos \theta} + e^{-jkh \cos \theta} \right] \sin \theta \\
\approx j \eta \frac{k l_o \Delta l e^{-jkr}}{2 \pi r} \left[ \cos(kh \cos \theta) \right] \sin \theta
\]

The normalized power pattern for the vertical infinitesimal dipole over a PEC ground is

\[
F(\theta) = \cos^2(kh \cos \theta) \sin^2 \theta
\]

\( h = 0.1 \lambda \)

\( h = 0.25 \lambda \)
\[ h = 0.5\lambda \]

\[ h = \lambda \]

\[ h = 2\lambda \]

\[ h = 10\lambda \]
Since the radiated fields of the infinitesimal dipole over ground are different from those of the isolated antenna, the basic parameters of the antenna are also different. The far fields of the infinitesimal dipole are:

\[ E_\theta \approx j \eta \frac{k I_o \Delta l e^{-jkr}}{2\pi r} [\cos(kh\cos\theta)] \sin\theta \]

\[ H_\phi = \frac{E_\theta}{\eta} \approx j \frac{k I_o \Delta l e^{-jkr}}{2\pi r} [\cos(kh\cos\theta)] \sin\theta \]

The time-average Poynting vector is:

\[ S = \frac{1}{2} \left[ E_\theta a_\theta \times H_\phi^* a_\phi \right] = \frac{|E_\theta|^2}{2\eta} a_r = S_r a_r \]

The corresponding radiation intensity function is:

\[ U(\theta,\phi) = r^2 S_r = \frac{\eta}{2} \left| \frac{k I_o \Delta l}{2\pi} \right|^2 \cos^2(kh\cos\theta) \sin^2\theta \]

\[ = \frac{\eta}{2} \left| \frac{I_o \Delta l}{\lambda} \right|^2 \cos^2(kh\cos\theta) \sin^2\theta \]

The maximum value of the radiation intensity function is found at \( \theta = \pi/2 \).

\[ [U(\theta,\phi)]_{\text{max}} = \frac{\eta}{2} \left| \frac{I_o \Delta l}{\lambda} \right|^2 \]

The radiated power is found by integrating the radiation intensity function:

\[ P_{rad} = \int_0^{2\pi} \int_0^{\pi/2} U(\theta,\phi) \sin\theta \, d\theta \, d\phi \]
The directivity function of the infinitesimal dipole over ground is

\[ P_{rad} = \frac{\eta}{2} \left| \frac{I_o \Delta l}{\lambda} \right|^2 \left[ 2 \pi \int_0^{\pi/2} \cos^2(kh \cos \theta) \sin^3 \theta \, d\theta \right] \]

\[ = \eta \pi \left| \frac{I_o \Delta l}{\lambda} \right|^2 \left[ \frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right] \]

\[ = \frac{1}{2} |I_o|^2 R_r \]

\[ R_r = 2 \pi \eta \left| \frac{\Delta l}{\lambda} \right|^2 \left[ \frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right] \]

(Infinitesimal dipole over ground radiation resistance)

The directivity function of the infinitesimal dipole over ground is

\[ D(\theta,\phi) = 4 \pi \frac{U(\theta,\phi)}{P_{rad}} = \frac{2 \cos^2(kh \cos \theta) \sin^2 \theta}{\left[ \frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right]} \]

so that the maximum directivity (at \( \theta = \pi/2 \)) is given by

\[ D_o = \frac{2}{\left[ \frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right]} \]

(Infinitesimal dipole over ground maximum directivity)
Given an infinitesimal dipole of length $\Delta l = \lambda/50$, we may plot the radiation resistance and maximum directivity as a function of the antenna height to see the effect of the ground plane.

For an isolated infinitesimal dipole of length $\Delta l = \lambda/50$, the radiation resistance is

$$R_r = 80\pi^2 \left( \frac{\Delta l}{\lambda} \right)^2 = 80\pi^2 \left( \frac{1}{50} \right)^2 = 0.316 \ \Omega$$

and the maximum directivity (independent of antenna length) is $D_o = 1.5$. Note that $R_r$ of the infinitesimal dipole over ground approaches twice that of $R_r$ for an isolated dipole as $h \to 0$ (see the relationship between a monopole antenna and its equivalent dipole antenna in the next section). As the height is increased, the radiation resistance of the infinitesimal dipole over ground approaches that of an isolated dipole. The directivity of the infinitesimal dipole over ground approaches a value twice that of the isolated dipole as $h \to 0$ and four times that of the isolated dipole as $h$ grows large. This follows from our definition of the total radiated power and maximum directivity for the isolated antenna and the antenna over ground.

$$P_{rad} = \frac{1}{2} |I_o|^2 R_r$$

$$D_o = 4\pi \frac{U_{\text{max}}}{P_{rad}}$$
First, we note the relationship between $U_{\text{max}}$ for the isolated dipole and the dipole over ground.

$$U_{\text{max}}^{\text{isolated}} = \frac{\eta}{8} \left| \frac{I_o \Delta l}{\lambda} \right|^2$$
$$U_{\text{max}}^{\text{over ground}} = \frac{\eta}{2} \left| \frac{I_o \Delta l}{\lambda} \right|^2$$

$$4U_{\text{max}}^{\text{isolated}} = U_{\text{max}}^{\text{over ground}}$$

Note that $U_{\text{max}}$ for the antenna over ground is independent of the height of the antenna over ground.

$h \rightarrow 0$

$$R_{r}^{\text{over ground}} = 2R_{r}^{\text{isolated}}$$
$$P_{\text{rad}}^{\text{over ground}} = 2P_{\text{rad}}^{\text{isolated}}$$
$$D_{o}^{\text{over ground}} = 4\pi \frac{U_{\text{max}}^{\text{over ground}}}{P_{\text{rad}}^{\text{over ground}}} = 4\pi \frac{4U_{\text{max}}^{\text{isolated}}}{2P_{\text{rad}}^{\text{isolated}}} = 2D_{o}^{\text{isolated}}$$

$h \rightarrow \text{large}$

$$R_{r}^{\text{over ground}} = R_{r}^{\text{isolated}}$$
$$P_{\text{rad}}^{\text{over ground}} = P_{\text{rad}}^{\text{isolated}}$$
$$D_{o}^{\text{over ground}} = 4\pi \frac{U_{\text{max}}^{\text{over ground}}}{P_{\text{rad}}^{\text{over ground}}} = 4\pi \frac{4U_{\text{max}}^{\text{isolated}}}{P_{\text{rad}}^{\text{isolated}}} = 4D_{o}^{\text{isolated}}$$
Using image theory, the monopole antenna over a PEC ground plane may be shown to be equivalent to a dipole antenna in a homogeneous region. The equivalent dipole is twice the length of the monopole and is driven with twice the antenna source voltage. These equivalent antennas generate the same fields in the region above the ground plane.
The input impedance of the equivalent antennas is given by

\[
(Z_{in})_{monopole} = \frac{V_{in}}{I_{in}} \quad (Z_{in})_{dipole} = \frac{2V_{in}}{I_{in}} = 2(Z_{in})_{monopole}
\]

The input impedance of the monopole is exactly one-half that of the equivalent dipole. Therefore, we may determine the monopole radiation resistance for monopoles of different lengths according to the results of the equivalent dipole.

Infinitesimal dipole
[length = \(\Delta l < \lambda/50\)]

\[R_r = 80\pi^2 \left( \frac{\Delta l}{\lambda} \right)^2\]

Infinitesimal monopole
[length = \(\Delta l < \lambda/100\)]

\[R_r = 40\pi^2 \left( \frac{\Delta l}{\lambda} \right)^2\]

Short dipole
[length = \(l, (\lambda/50 \leq l \leq \lambda/10)\)]

\[R_r = 20\pi^2 \left( \frac{\Delta l}{\lambda} \right)^2\]

Short monopole
[length = \(l, (\lambda/100 \leq l \leq \lambda/20)\)]

\[R_r = 10\pi^2 \left( \frac{\Delta l}{\lambda} \right)^2\]

Lossless half-wave dipole
[length = \(l = \lambda/2\)]

\[Z_{in} = (73.14 + j42.5)\Omega\]

Lossless quarter-wave monopole
[length = \(l = \lambda/4\)]

\[Z_{in} = (36.57 + j21.25)\Omega\]
The total power radiated by the monopole is one-half that of the equivalent dipole. But, the monopole radiates into one-half the volume of the dipole yielding equivalent fields and power densities in the upper half space.

\[
\langle P_{rad} \rangle_{\text{monopole}} = \frac{1}{2} \langle P_{rad} \rangle_{\text{dipole}}
\]

\[
\int \int [U(\theta, \phi)]_{\text{monopole}} \sin \theta \, d\theta \, d\phi = \frac{1}{2} \int \int [U(\theta, \phi)]_{\text{dipole}} \sin \theta \, d\theta \, d\phi
\]

\[
[U(\theta, \phi)]_{\text{monopole}} = [U(\theta, \phi)]_{\text{dipole}}
\]

The directivities of the two equivalent antennas are related by

\[
[D(\theta, \phi)]_{\text{monopole}} = 4\pi \left( \frac{[U(\theta, \phi)]_{\text{monopole}}}{\langle P_{rad} \rangle_{\text{monopole}}} \right)
\]

\[
= 4\pi \left( \frac{1}{2} [U(\theta, \phi)]_{\text{dipole}} \right)
\]

\[
= 2 [D(\theta, \phi)]_{\text{dipole}}
\]

Infinitesimal dipole
[length = \Delta l < \lambda/50] \quad D_o = 1.5 \ (1.76 \text{ dB})

Infinitesimal monopole
[length = \Delta l < \lambda/100] \quad D_o = 3 \ (4.77 \text{ dB})

Lossless half-wave dipole
[length = l = \lambda/2] \quad D_o = 1.641 \ (2.15 \text{ dB})

Lossless quarter-wave monopole
[length = l = \lambda/4] \quad D_o = 3.282 \ (5.16 \text{ dB})
Ground Effects on Antennas

At most frequencies, the conductivity of the earth is such that the ground may be accurately approximated by a PEC. Given an antenna located over a PEC ground plane, the radiated fields of the antenna over ground can be determined easily using image theory. The fields radiated by the antenna over a PEC ground excite currents on the surface of the ground plane which re-radiate (scatter) the incident waves from the antenna. We may also view the PEC ground plane as a perfect reflector of the incident EM waves. The direct wave/reflected wave interpretation of the image theory results for the infinitesimal dipole over a PEC ground is shown below.

\[ E_\theta \approx j \eta \frac{k I_o \Delta l}{4 \pi r} \left[ e^{-j k (r - h \cos \theta)} + e^{-j k (r + h \cos \theta)} \right] \sin \theta \]

direct wave \hspace{1cm} reflected wave
At lower frequencies (approximately 100 MHz and below), the electric fields associated with the incident wave may penetrate into the lossy ground, exciting currents in the ground which produce ohmic losses. These losses reduce the radiation efficiency of the antenna. They also effect the radiation pattern of the antenna since the incident waves are not perfectly reflected by the ground plane. Image theory can still be used for the lossy ground case, although the magnitude of the reflected wave must be reduced from that found in the PEC ground case. The strength of the image antenna in the lossy ground case can be found by multiplying the strength of the image antenna in the PEC ground case by the appropriate plane wave reflection coefficient for the proper polarization ($\Gamma_v$).

\[
E_\theta \approx j \eta \Delta l e^{-jkr} \frac{k I_o}{4\pi r} e^{jkh\cos\theta} \sin\theta \left[ e^{jkh\cos\theta} + \Gamma_v e^{-jkh\cos\theta} \right]
\]

\[
\Gamma_v = \frac{\eta_o \cos \theta - \eta_1 \cos \theta}{\eta_o \cos \theta + \eta_1 \cos \theta}
\]

\[
\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}}, \quad \eta_1 = \sqrt{\frac{j\omega \mu_1}{\sigma_1 + j\omega \epsilon_1}}
\]
If we plot the radiation pattern of the vertical dipole over ground for cases of a PEC ground and a lossy ground, we find that the elevation plane pattern for the lossy ground case is tilted upward such that the radiation maximum does not occur on the ground plane but at some angle tilted upward from the ground plane (see Figure 4.28, p. 183). This alignment of the radiation maximum may or may not cause a problem depending on the application. However, if both the transmit and receive antennas are located close to a lossy ground, then a very inefficient system will result. The antenna over lossy ground can be made to behave more like an antenna over perfect ground by constructing a ground plane beneath the antenna. At low frequencies, a solid conducting sheet is impractical because of its size. However, a system of wires known as a radial ground system can significantly enhance the performance of the antenna over lossy ground.

Monopole with a radial ground system

The radial wires provide a return path for the currents produced within the lossy ground. Broadcast AM transmitting antennas typically use a radial
ground system with 120 quarter wavelength radial wires (3° spacing).

The reflection coefficient scheme can also be applied to horizontal antennas above a lossy ground plane. The proper reflection coefficient must be used based on the orientation of the electric field (parallel or perpendicular polarization).

The Effect of Earth Curvature

Antennas on spacecraft and aircraft in flight see the same effect that antennas located close to the ground experience except that the height of the antenna over the conducting ground means that the shape of the ground (curvature of the earth) can have a significant effect on the scattered field. In cases like these, the curvature of the reflecting ground must be accounted for to yield accurate values for the reflected waves.

Antennas in Wireless Communications

Wire antennas such as dipoles and monopoles are used extensively in wireless communications applications. The base stations in wireless communications are most often arrays (Ch. 6) of dipoles. Hand-held units such as cell phones typically use monopoles. Monopoles are simple, small, cheap, efficient, easy to match, omnidirectional (according to their orientation) and relatively broadband antennas. The equations for the performance of a monopole antenna presented in this chapter have assumed that the antenna is located over an infinite ground plane. The monopole on the hand-held unit is not driven relative to the earth ground but rather (a.) the conducting case of the unit or (b.) the circuit board of the unit. The resonant frequency and input impedance of the hand-held monopole are not greatly different than that of the monopole over a infinite ground plane. The pattern of the hand-held unit monopole is different than that of the monopole over an infinite ground plane due to the different distribution of currents. Other antennas used on hand-held units are loops (Ch. 5), microstrip (patch) antennas (Ch. 14) and the planar inverted F antenna (PIFA). In wireless applications, the antenna can be designed to
perform in a typical scenario, but we cannot account for all scatterer geometries which we may encounter (power lines, buildings, etc.). Thus, the scattered signals from nearby conductors can have an adverse effect on the system performance. The detrimental effect of these unwanted scattered signals is commonly referred to as *multipath*. 