Solution

1. (a) We have the differential equation
\[
\frac{d}{dt} y(t) + 2y(t) = u(t)
\]
with initial condition
\[y(0^-) = 0.\]
Taking the Laplace transform of the differential equation yields
\[
sY(s) - y(0^-) + 2Y(s) = \frac{1}{s}
\]
\[
(s + 2)Y(s) = \frac{1}{s}
\]
\[
Y(s) = \frac{1}{s(s + 2)}.
\]
To invert the Laplace transform, we do partial fraction expansion,
\[
Y(s) = \frac{1}{s(s + 2)} = \frac{K_1}{s} + \frac{K_2}{s + 2}
\]
\[
K_1 = sY(s) \bigg|_{s=0} = \frac{1}{s + 2} \bigg|_{s=0} = \frac{1}{2}
\]
\[
K_2 = (s + 2)Y(s) \bigg|_{s=-2} = \frac{1}{s} \bigg|_{s=-2} = -\frac{1}{2}
\]
The inverse Laplace transform is then
\[
y(t) = \left[ \frac{1}{2} - \frac{1}{2} e^{-2t} \right] u(t).
\]
(b) We have the differential equation
\[
\frac{d}{dt} y(t) - 2y(t) = u(t)
\]
with initial condition
\[y(0^-) = 1.\]
Taking the Laplace transform of the differential equation yields
\[
sY(s) - y(0^-) - 2Y(s) = \frac{1}{s}
\]
\[
(s - 2)Y(s) = \frac{1}{s + 1}
\]
\[
Y(s) = \frac{s + 1}{s(s - 2)}.
\]
Partial fraction expansion:

\[ Y(s) = \frac{s + 1}{s(s - 2)} = \frac{K_1}{s} + \frac{K_2}{s - 2} \]

\[ K_1 = sY(s) \bigg|_{s=0} = s + 1 \bigg|_{s=0} = \frac{1}{2} \]

\[ K_2 = (s - 2)Y(s) \bigg|_{s=2} = \frac{s + 1}{s} \bigg|_{s=2} = \frac{3}{2} \]

The inverse Laplace transform is then

\[ y(t) = \left[ \frac{3}{2}e^{2t} - \frac{1}{2} \right] u(t). \]

(c) We have the differential equation

\[ \frac{d^2}{dt^2} y(t) + 6 \frac{dy}{dt} + 8y(t) = u(t) \]

with initial conditions

\[ y(0^-) = 0, \quad y'(0^-) = 1. \]

Taking the Laplace transform of the differential equation yields

\[ \left[ s^2Y(s) - sy(0^-) - y'(0^-) \right] + 6 \left[ sY(s) - y(0^-) \right] + 8Y(s) = \frac{1}{s} \]

\[ (s^2 + 6s + 8)Y(s) = \frac{1}{s} + 1 \]

\[ Y(s) = \frac{s + 1}{s(s^2 + 6s + 8)} = \frac{s + 1}{s(s + 2)(s + 4)} \]
Partial fraction expansion:

\[ Y(s) = \frac{s + 1}{s(s + 2)(s + 4)} = \frac{K_1}{s} + \frac{K_2}{s + 2} + \frac{K_3}{s + 4} \]

\[ K_1 = sY(s) \bigg|_{s=0} = \frac{s + 1}{(s + 2)(s + 4)} \bigg|_{s=0} = \frac{1}{8} \]

\[ K_2 = (s + 2)Y(s) \bigg|_{s=-2} = \frac{s + 1}{s(s + 4)} \bigg|_{s=-2} = \frac{1}{4} \]

\[ K_3 = (s + 4)Y(s) \bigg|_{s=-4} = \frac{s + 1}{s(s + 2)} \bigg|_{s=-4} = -\frac{3}{8} \]

The inverse Laplace transform is

\[ y(t) = \left[ \frac{1}{8} + \frac{1}{4}e^{-2t} - \frac{3}{8}e^{-4t} \right] u(t). \]

(d) We have the differential equation

\[ \frac{d^2}{dt^2} y(t) + 6 \frac{dy}{dt} y(t) + 9y(t) = \sin(2t) u(t) \]

with initial conditions

\[ y(0^-) = 0, \quad y'(0) = 0. \]

The Laplace transform of the differential equation yields

\[ \left[ s^2Y(s) - sy(0^-) - y'(0^-) \right] + 6 \left[ sY(s) - y(0^-) \right] + 9Y(s) = \frac{2}{s^2 + 4} \]

\[ (s^2 + 6s + 9)Y(s) = \frac{2}{s^2 + 4} \]

\[ Y(s) = \frac{2}{(s^2 + 6s + 9)(s^2 + 4)} \]

\[ Y(s) = \frac{2}{(s + 3)^2(s^2 + 4)} \]
Partial fraction expansion:

\[ Y(s) = \frac{2}{(s + 3)^2(s^2 + 4)} = \frac{K_1}{s + 3} + \frac{K_2}{(s + 3)^2} + \frac{K_3}{s - (j2)} + \frac{K_3^*}{s - (-j2)} \]

\[ K_1 = \frac{\frac{d}{ds}(s + 3)^2Y(s)}{s = -3} = \frac{d}{ds} \frac{2}{s^2 + 4} \bigg|_{s = -3} = \frac{-4s}{(s^2 + 4)^2} \bigg|_{s = -3} = \frac{12}{169} \]

\[ K_2 = \frac{(s + 3)^2Y(s)}{s = -3} = \frac{2}{s^2 + 4} \bigg|_{s = -3} = \frac{2}{13} \]

\[ K_3 = \frac{(s - (j2))Y(s)}{s = j2} = \frac{2}{(s + 3)^2(s - (-j2))} \bigg|_{s = j2} = \frac{2}{(3 + j2)^2(j4)} \approx 0.0385 - 157.4^\circ \]

The inverse Laplace transform is

\[ y(t) \approx \left[ \frac{12}{169}e^{-3t} + \frac{2}{13}te^{-3t} + 0.077\cos(2t - 157.4^\circ) \right] u(t). \]

2. The circuit transformed into the s-domain is:
The equivalent impedance of the circuit is

\[
Z_{eq} = \left( R + \frac{1}{sC} \right) + R \left\| \frac{1}{sC} \right.
\]

\[
= R + \frac{1}{sC} + \frac{R}{sC}
\]

\[
= \frac{sRC + 1}{sC} + \frac{R}{sRC + 1}
\]

\[
= \frac{s^2R^2C^2 + 3RC + 1}{sC(sRC + 1)}
\]

The output, via voltage division, is

\[
Y(s) = \frac{(R + \frac{1}{sC})}{Z_{eq}} X(s)
\]

\[
= \frac{sRC + 1}{sC} \cdot \frac{sC(sRC + 1)}{s^2R^2C^2 + 3RC + 1} X(s)
\]

\[
= \frac{s^2R^2C^2 + 3RC + 1}{s^2R^2C^2 + 3RC + 1} X(s)
\]

\[
\therefore H(s) = \frac{Y(s)}{X(s)}
\]

\[
= \frac{s^2R^2C^2 + 3RC + 1}{s^2R^2C^2 + 3RC + 1}
\]

3. (a) The system block diagram is

where a block of a gain of 1 has been explicitly included in the feedback loop. Reducing the series connection in the upper path produces

\[
\frac{10(s + 1)}{s} \cdot \frac{1}{(s + 1)(s + 2)} = \frac{10}{s(s + 2)}.
\]

Reducing the feedback loop produces

\[
\frac{10}{s(s + 2)} = \frac{10}{s + 2}.
\]
(b) The system block diagram is

Reducing the parallel and series connections of the upper branch yields

\[
\left[ 10(s + 9) + \frac{80}{s} \right] \cdot \frac{1}{(s + 1)(s + 3)} = \frac{10s^2 + 90s + 80}{s} \cdot \frac{1}{(s + 1)(s + 3)} = \frac{10s + 8}{s(s + 3)}
\]

Reducing the feedback loop produces

\[
\frac{10(s + 8)}{s(s + 3)} \cdot \frac{20}{s + 20} = \frac{10(s + 8)(s + 20)}{s(s + 3)(s + 20) + 200(s + 8)} = \frac{10s^2 + 280s + 1600}{s^3 + 23s^2 + 60s + 200s + 1600} = \frac{10s^2 + 280s + 1600}{s^3 + 23s^2 + 260s + 1600}
\]
Finally, including the gain, we have
\[ Y(s) = \frac{100s^2 + 2800s + 16000}{s^3 + 23s^2 + 260s + 1600} \]

(c) The system block diagram is

Combining the series connection in the middle yields
\[ Y(s) = \frac{4}{(s+2)(s+10)^2} \]

Reducing the inner feedback loop yields
\[ Y(s) = \frac{4}{s^2 + 20s + 20} \]
Finally, reducing the parallel connection yields

\[
(s + 2) + \frac{4}{s^2 + 20s + 20} = \frac{s^3 + 22s^2 + 60s + 44}{s^2 + 20s + 20}
\]

\[X(s) \xrightarrow{\frac{s^3 + 22s^2 + 60s + 44}{s^2 + 20s + 20}} Y(s)\]