CHAPTER 5. AC CIRCUIT ANALYSIS and the FREQUENCY DOMAIN

5.1 SINUSOIDAL SIGNALS

Signals with natural origins are of the form of vibrational waves. Vibrational waves are a means by which energy is transferred from one point to another without transfer of any material. If a string is plucked its vibrations will create ripples in the fabric of the gas or liquid surrounding it and these ripples will propagate or radiate from the string to a distant ear or shore. If an electromagnetic field is plucked it will create ripples in the fabric of space-time and they will propagate or radiate to a distant receptor, some of which are constructs, some of which are not. And although electromagnetic field vibrations exist in many natural forms, the ones that are of interest to the electronics venue are those at the frequency levels that are consistent with the response(s) of electric circuits.

All waves are characterized by frequency, amplitude, and relative phase shifts. If transmission distances are included then the wavelength and velocity of propagation are also important. But in their interaction with circuit topologies only the parameters represented by equation (5.1-1) will matter.

\[ V(t) = V_b \sin(\omega t + \phi) \]  

for which \( \omega \) is the radian frequency and is related to the harmonic frequency \( f \) by

\[ f = \omega / 2\pi \approx 0.16\omega \]  

(Use this approximation as a quick assessment) Figure 5.1-1 shows sinusoidal electromagnetic waves in the form of current and voltage. In this case the current is lagging behind that of the voltage by approximately 90°, which is a phenomenon peculiar to circuits dominated by inductance.

![Figure 5.1-1. Pspice traces. The current is the (red) dashed-line trace (current lags the voltage).](image)

Sinusoidal signals may also be represented by the complex arithmetic (Euler) form, i.e.

\[ V(t) = V_b e^{j\omega t} e^{j\phi} \]  

for which the mathematics is usually easier, friendlier, and more tractable.
5.2 CIRCUIT RESPONSE to SINUSOIDAL SIGNALS

Reactive components are components which react to the $I(t)$ and $V(t)$ changes with respect to time. There are two such components that do so, the capacitance and the inductance. Since these components store energy it takes (reactive) time for them to charge and discharge, as represented by:

$$I(t) = C \frac{dV}{dt} \quad \text{(5.2-1a)}$$
$$V(t) = L \frac{dI}{dt} \quad \text{(5.2-1b)}$$

Since time derivatives are involved, a circuit network with reactive components will have an output which is a sum of sine and cosine terms. And such is represented by figure 5.2-1.

![Figure 5.2-1. Circuit response to sinusoidal input.](image)

Sine and cosine are complementary functions with phase difference of 90°. A sum such as that at the output port may be reduced to a more tractable form by means of a little algebra and trigonometry, i.e.

$$v_o = A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \cos \omega t + \frac{B}{\sqrt{A^2 + B^2}} \sin \omega t \right)$$

$$= \sqrt{A^2 + B^2} \left( \cos \omega t \cos \phi + \sin \omega t \sin \phi \right)$$

$$= \sqrt{A^2 + B^2} \cos(\omega t - \phi) \quad \text{(5.2-2)}$$

where the phase shift is $\phi = \tan^{-1}(B/A)$

$$\quad \quad \text{(5.2-3)}$$

Since trigonometry and geometry are mathematical cousins the relationships represented by equations (5.2-2) and (5.2-3) may also be identified by figure 5.2-2.

![Figure 5.2-2. Plane trigonometry](image)
EXAMPLE 5.2-1: \[ v_o = -6 \cos \omega t + 8 \sin \omega t = \sqrt{6^2 + 8^2} \cos(\omega t - 127°) \]

And then \[ \phi = \tan^{-1}(8/(-6)) = 127° \] (Note second quadrant condition)

The rest of the story is that a circuit forms a sum of terms, either by KVL or KCL. And with reactive components in the picture then some of these terms include derivatives. For a simple loop and a single reactive component, a relatively straightforward outcome results, as illustrated by example 5.2-2.

EXAMPLE 5.2-2: Evaluate the circuit shown and determine output in the form \[ V_O \cos(\omega t - \phi). \]

SOLUTION: Using KVL

\[ L \frac{di}{dt} + iR = (v_s = V_m \cos \omega t) \]

Assume: \[ i = A \cos \omega t + B \sin \omega t \]
Then \[ L \frac{di}{dt} = -A \omega L \sin \omega t + B \omega L \cos \omega t \]

Equating coefficients in the KVL sum

\[
\begin{align*}
\text{sin } \omega t: & \quad -A \omega L + RB = 0 \\
\text{cos } \omega t: & \quad AR + \omega LB = V_m
\end{align*}
\]

eliminate \( A \): \[ B(R^2 + \omega^2 L^2) = \omega L V_m \] \[ \rightarrow \quad B = \omega L V_m / (R^2 + \omega^2 L^2) \]

eliminate \( B \): \[ A(\omega^2 L^2 + R^2) = RV_m \] \[ \rightarrow \quad A = RV_m / (R^2 + \omega^2 L^2) \]

and therefore \[ \phi = \tan^{-1}(B/A) = \tan^{-1}(\omega L/R) \] and \[ \sqrt{A^2 + B^2} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} = \frac{V_m}{|Z|} \]

where \[ |Z| = \sqrt{R^2 + \omega^2 L^2} \] is of the form of a reactive resistance

The outcome is then \[ i = \frac{V_m}{|Z|} \cos(\omega t - \phi) \] for which \[ v_o = iR = V_m \frac{R}{|Z|} \cos(\omega t - \phi) \]

Attention should be called to the fact that the RL time constant \( \tau_L = L/R \) emerged from the analysis of this RL example. This outcome should not be unexpected, and simplifies the results to:

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This aspect is specific to networks characterized by a single time constant. Consider an example using KCL and a capacitance component $C$ as represented by example 5.2-3.

EXAMPLE 5.2-3: Evaluate the circuit shown and determine output in the form $V_O \cos (\omega t - \phi)$.

SOLUTION: Using KCL

$$i_s = I_m \cos \omega t = C \frac{dv}{dt} + G v$$

Assume that the voltage at the output node is of the form

$$v = A \cos \omega t + B \sin \omega t$$

for which

$$C \frac{dv}{dt} = -A \omega C \sin \omega t + B \omega C \cos \omega t$$

Equating coefficients in the KCL sum

$$\sin \omega t : \quad -A \omega C + GB = 0$$

$$\cos \omega t : \quad AG + \omega CB = I_m$$

eliminating $A$: \hspace{1cm} $B = I_m \omega C / (G^2 + \omega^2 C^2)$

eliminating $B$: \hspace{1cm} $A = I_m G / (G^2 + \omega^2 C^2)$

and therefore

$$\phi = \tan^{-1} \left( \frac{B}{A} \right) = \tan^{-1} \left( \frac{\omega C}{G} \right) = \tan^{-1} \left[ \omega RC \right]$$

and

$$\sqrt{A^2 + B^2} = \frac{I_m}{\sqrt{G^2 + \omega^2 C^2}} \equiv \frac{I_m}{|Y|}$$

The voltage at the output node then becomes:

$$v = \frac{I_m}{|Y|} \cos (\omega t - \phi)$$

where $|Y| = \sqrt{G^2 + \omega^2 C^2}$ is of the form of a reactive conductance

Similar to its cousin (example 5.2-2) the time constant of this topology, $\tau_c = C/G = RC$, is a factor in its time-dependent sinusoidal response, i.e.

$$\phi = \tan^{-1} \left[ \frac{\omega RC}{G} \right] = \tan^{-1} \left[ \omega \tau_c \right]$$

$$|Y| = \sqrt{G^2 + \omega^2 C^2} = G \sqrt{1 + \omega^2 (C/G)^2} = G \sqrt{1 + \omega^2 \tau^2}$$
Also: In the matter of phase shifts it should be noted that

**Capacitance circuit:** (example 5.2-3): \( i \) leads \( v \)

**Inductance circuit:** (example 5.2-2): \( i \) lags \( v \)

Large inductances and capacitances are representative of the power grid and the lead-lag context is an essential part of the power grid vocabulary.

### 5.3 CIRCUIT ANALYSIS IN THE COMPLEX PLANE

Sinusoidal analysis at the trigonometric level is a mathematical exercise that requires discipline and overhead. Life and mathematics is far easier, completely equivalent, and more informative when carried out in the complex plane. Complex numbers are of the form

\[
z = x + jy = (r \cos \phi) + j(r \sin \phi) = re^{j\phi}
\]  

(5.3-1)

Magnitude and phase of a complex number are

\[
r = \sqrt{x^2 + y^2}
\]  

(5.3-2a)

\[
\phi = \tan^{-1}(y/x)
\]  

(5.3-2b)

These entirely consistent with the nomenclature represented by figure 5.2-2. The use and context appropriate to circuit analysis is shown by figure 5.3-1.

**Figure 5.3-1.** Linear reactive circuits in the complex plane. Components in series.
The real part is represented as \( R = |Z| \cos \phi \)
The imaginary part is \( X = |Z| \sin \phi \)
And the phase angle is \( \phi = \tan^{-1} \left( \frac{X}{R} \right) \)

The magnitude and phase are as indicated by the figure. Circuit constructs in the complex plane, particularly a series string of components, are characterized by the complex form

\[ Z = R + jX \]

(5.3-3)
as if the construct were a single reactive resistance. Equation (5.3-3) is defined as impedance. The real part of the impedance = \( R \) is either resistance or a passive equivalent. Magnitude \( X \) is the imaginary part of the impedance and is due to reactive components within the circuit. And consequently \( X \) is defined as the reactance.

If circuit components are in parallel they will add in the like manner as conductances except in the complex domain. The complex number result is characterized as

\[ Y = G + jB \]

(5.3-4)
Equation (5.3-4) is defined as admittance. The real part of the admittance = \( G \) and will be either the conductance or a passive equivalent. Magnitude \( B \) is the imaginary part of the admittance and is due to the reactive components within the circuit. \( B \) is defined as the susceptance.

A circuit construct in the complex plane can be formulated as either an admittance or as an impedance.

Time dependence of reactive components within a circuit is reflected by the same differential forms as before except that the electrical signals \( v(t) \) or \( i(t) \) are in terms of a complex form, i.e.

\[ x(t) = x_a e^{j\omega t} \]

(5.3-5)
and equations (5.2-1a) and (5.2-1b) will give reactive responses to current and voltage signals of

\[ I(t) = I_a e^{j\omega t} = C \frac{d}{dt} \left( V_m e^{j\omega t} \right) = j\omega C V_m e^{j\omega t} \]

for which the effect of the capacitance within the circuit may then be represented as

\[ I_m = j\omega C \times V_m \]

(5.3-6a)
Similarly for the inductance

\[ V(t) = V_m e^{j\omega t} = L \frac{d}{dt} \left( I_m e^{j\omega t} \right) = j\omega L I_m e^{j\omega t} \]
for which the effect of the inductance in the circuit may then be represented as

\[ V_m = j \omega L \times I_m \]  \hspace{1cm} (5.3-6b)

As might be expected, a reconsideration of example 5.2-2 (as example 5.3-1) with use of complex variables would be a more efficient process since there is no need to separately resolve \( \sin(x) \) and \( \cos(x) \) terms.

As a confirmation check, example (5.3-1) (below) is undertaken as a repeat of example (5.2-2) except applying complex variables for \( i \) and \( v \).

**EXAMPLE 5.3-1:** Evaluate the circuit shown and determine current \( i(t) \) through resistance \( R \) using the complex variable formulation

**SOLUTION:** Let \( v_S(t) = V_m e^{j \omega t} \)
and \( i(t) = I_m e^{j \omega t} \)

and apply KVL, for which:

\[ V_m e^{j \omega t} = L \frac{di}{dt} + iR = (j \omega L + R) I_m e^{j \omega t} \]

Dividing out the \( e^{j \omega t} \) factor gives

\[ V_m = (j \omega L + R) I_m \]

The complex coefficient \((j \omega L + R)\) can be restated as its Euler equivalent, for which:

\[ V_m = \left( \sqrt{R^2 + \omega^2 L^2} e^{j \phi} \right) I_m = |Z| e^{j \phi} I_m \]

The result is then

\[ V_m = Z I_m \]

where \( Z = \text{impedance} = j \omega L + R = |Z| e^{j \phi} \)

with \( |Z| = \sqrt{R^2 + \omega^2 L^2} \) and \( \phi = \tan^{-1}(\omega L/R) = \tan^{-1}(\omega R) \)

= same as equations (5.2-4b) and (5.2-4a) that emerged from the RL topology of example (5.2-2)

The current though \( R \) is then

\[ i(t) = I_m e^{j \omega t} = \frac{V_m}{Z} e^{j \omega t} = \frac{V_m}{|Z|} e^{-j \phi} \times e^{j \omega t} \]
Similarly, reconsidering example 5.2-3 as example 5.3-2 and analyzing the circuit by means of complex variables gives a similar reduced process.

**EXAMPLE 5.3-2:** Evaluate the circuit shown and determine output \( v(t) \).

**SOLUTION:** Using \( i_s(t) = I_m e^{j\omega t} \) and \( v(t) = V_m e^{j\omega t} \)

and applying KCL for which

\[
I_m e^{j\omega t} = C \frac{dv}{dt} + vG = (j\omega C + G) V_m e^{j\omega t}
\]

Dividing out the common factor \( e^{j\omega t} \) results in \( I_m = (j\omega C + G) V_m = Y V_m \)

For which \( Y \) is the (complex) admittance \( (j\omega C + G) \)

\( Y \) may also be written as

\[
Y = |Y| e^{j\phi} \quad \text{with (complex number) magnitude} \quad |Y| = \sqrt{G^2 + \omega^2 C^2}
\]

and phase shift between input and output of \( \phi = \tan^{-1}(\omega RC) = \tan^{-1}(\omega \tau_C) \)

where the time constant \( \tau_C \) is a not unexpected consequence of the RC circuit topology

Complex voltage across \( R \) is then:

\[
v(t) = V_m e^{j\omega t} = \frac{I_m}{Y} e^{j\omega t} = \frac{I_m}{|Y|} e^{-j\phi} \times e^{j\omega t}
\]

The use of lower case \( i \) and \( v \) in these examples reflect an analysis that identifies with signal voltages and currents as well as the analysis for fixed-frequency systems such as the power grid.

These examples also suggest that it is appropriate to adopt a reactive form of Ohm’s law that accommodates AC signals and the \( L \) and \( C \) circuit components. The appropriate form is represented by

\[
i = Yv \quad \text{(5.3-7a)}
\]

\[
v = Zi \quad \text{(5.3-7b)}
\]

for which \( Z = \text{impedance} \) and \( Y = \text{admittance} \).
This implication also identifies that $Z$ and $Y$ have the same reciprocal relationship as resistance and conductance, i.e.

$$Y = 1/Z$$  \hspace{1cm} (5.3-8)

These relationships entirely fulfill the tenets of linear circuit analysis developed for resistance networks. Therefore it is safe to assume the same rules that were developed thereto, e.g.

$$\sum_{node} i_k = 0$$ \hspace{1cm} (5.3-9)

for Kirchoff’s current law (KCL). The lower case usage is implication of $i_k$ as signal current(s), but not exclusively so. The convention of signal current out of a node as positive and current into the node as negative is retained. A branch is any type of component, whether passive or reactive. For linear components these are $R$, $L$ and $C$. Other components will also obey the same conventions.

Likewise for Kirchoff’s voltage law

$$\sum_{loop} v_k = 0$$ \hspace{1cm} (5.3-10)

It recognizes that the sum of voltage increments and decrements around each loop must add to zero, whether or not the components are passive or active.

Whether (5.3-9) or (5.3-10) is applied a system of equations results. The most likely option is nodal analysis as represented by equation (5.3-11)

$$Y_{11}v_1 + Y_{12}v_2 + \cdots + Y_{1n}v_n = i_{s1}$$

$$Y_{21}v_1 + Y_{22}v_2 + \cdots + Y_{2n}v_n = i_{s2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$Y_{n1}v_1 + Y_{n2}v_2 + \cdots + Y_{nn}v_n = i_{sn}$$  \hspace{1cm} (5.3-11)

Similarly for loop analysis.
EXAMPLE 5.3-3a: Consider the circuit network of figure 5.3-2, which has three unresolved nodes. Two of the branches include capacitances and if KCL is applied a nodal admittance analysis of the form reflected by equation (5.3-11) will result.

Figure 5.3-2. Example network with a mix of $R$ and $C$ branch components.

Nodal analysis at node $V_A$ gives

$$(G_S + G_I + G_4 + j\omega C_1)V_A - (G_I + j\omega C_1)V_B - G_4V_C - G_SV_S = 0$$  \hspace{1cm} (5.3-12a)

Nodal analysis at $V_B$ gives

$$(G_I + G_2 + G_3 + j\omega C_1 + j\omega C_3)V_B - (G_I + j\omega C_1)V_A - (G_3 + j\omega C_3)V_C = 0$$  \hspace{1cm} (5.3-12b)

Nodal analysis at $V_C$ gives

$$(G_4 + G_3 + G_5 + j\omega C_3)V_C - (G_3 + j\omega C_3)V_B - G_4V_A = 0$$  \hspace{1cm} (5.3-12c)

If these are arranged in matrix (linear algebra) form then

$$\begin{bmatrix} G_S + G_I + G_4 + j\omega C_1 & -(G_I + j\omega C_1) & -G_4 \\ -(G_I + j\omega C_3) & G_I + G_2 + G_3 + j\omega(C_1 + C_3) & -(G_I + j\omega C_1) \\ -G_4 & -(G_I + j\omega C_1) & G_4 + G_3 + G_5 + j\omega C_3 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} G_SV_S \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (5.3-13)

Equation (5.3-13) indicates why circuit simulation software might be a better alternative than a grit-and-grind hand analysis.

If the same exercise is redeployed with values (which, looking ahead, is example 5.3-3) then there might be some simplification. But don’t expect much. Complex arithmetic is not for wimps.
EXAMPLE 5.3-3b: Render the circuit shown into its KCL linear algebraic form (i.e. equation (5.3-10)) using nodal analysis.

SOLUTION:

This example is the same as figure 5.3-2 except with values. From the figure the radian frequency is

\[ \omega = 2\pi f = 2\pi \times 32\text{Hz} \approx 200\text{r/s} = 0.2\text{kr/s} \]

Hence equation (5.3-12a) becomes

\[(1 + 0.5 + 1 + 2.0j)V_A - (0.5 + 2.0j)V_B - 1.0V_C = 5.0 \quad \text{(values in m\(\Omega\) or in V)} \]

Equation (5.3-12b) becomes

\[(0.5 + 2 + 1 + j(2.0 + 1.0))V_B - (0.5 + 2.0j)V_A - (1 + 1.0j)V_C = 0 \]

Equation (5.3-12c) becomes

\[(1 + 1 + 0.1 + 1.0j)V_C - (1 + 1.0j)V_B - 1.0V_A = 0 \]

where all admittance values are in m\(\Omega\)

So the rendering as a linear algebraic form is

\[
\begin{bmatrix}
2.5 + 2j & -0.5 - 2j & -1 \\
-(0.5 + 2j) & 3.5 + 3j & -(1 + j) \\
-1 & -(1 + j) & 2.1 + j
\end{bmatrix}
\begin{bmatrix}
V_A \\
V_B \\
V_C
\end{bmatrix}
= \begin{bmatrix}
5.0 \\
0 \\
0
\end{bmatrix}
\]

After resolving the complex arithmetic, node voltages \(V_A\), \(V_B\) and \(V_C\), with respective phase shifts will result. But the process requires a fair amount of overhead in complex arithmetic, a level of mathematics more appropriate to use of a software utility than to the pencil and paper arithmetic.

Other analytical techniques that were appropriate to resistance networks are equally applicable. Thevenin and Norton Theorems will identify an equivalent output (Thevenin) impedance \(Z_{th}\) or an equivalent (Norton) admittance \(Y_n\). Matching load conditions will also obey the same specification as identified in chapter 2 and play a significant role in UHF circuit analysis.
5.4 FREQUENCY RESPONSE OF NETWORKS

The effect of sinusoidal signals on reactive components is summarized by

\[ Z_L = j\omega L \] (5.4-1a)

\[ Y_C = j\omega C \] (5.4-1b)

These relationships identify the specific impedance/admittance equivalents appropriate to inductances and capacitances as circuit components. If necessary, either can also be expressed in terms of its reciprocal form as implied by equation (5.3-8). For example, the impedance of a capacitance is

\[ Z_C = \frac{1}{Y_C} = \frac{1}{j\omega C} = -\frac{j}{\omega C} \] (5.4-2)

Equation (5.4-2) suggests that complex fractions should be rendered so the real and imaginary parts are arithmetically separable in order to accommodate a summation of terms.

**RLC circuits** will include time constants specific to \( L \) and \( C \) of the form

\[ \tau_C = C/G = RC \] (5.4-3a)

\[ \tau_L = L/R \] (5.4-3b)

Time constants of these forms define the frequency response of a circuit.

For example, consider a simple \( RC \) circuit with evaluation in both analytical form and in pspice:

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**EXAMPLE 5.4-1:** (a) Determine the transfer function \( v_1/v_I = T(\omega) \) as a function of frequency and (b) render the circuit in pspice and compare to the analytical formulation.

**SOLUTION:** (a) Nodal analysis at \( V_I \) and use of equation (5.4-1b) gives

\[ v_I (G_1 + j\omega C) - v_S G_1 = 0 \]

For which

\[ T(\omega) = \frac{v_I}{v_S} = \frac{G_1}{G_1 + j\omega C} = \frac{1}{1 + j\omega C_1/G_1} = \frac{1}{1 + j\omega \tau_1} \]
The magnitude $|T(\omega)|$ is defined by the magnitudes of the complex numbers. In this case the only complex number is that of the denominator. Therefore

$$\frac{|v_i|}{|v_s|} = \frac{1}{\sqrt{1 + (\omega \tau_i)^2}} \quad \text{(E5.4-1a)}$$

The phase shift associated with this transfer function is

$$\phi = -\tan^{-1}(\omega \tau_i) \quad \text{(E5.4-1b)}$$

where the negative is a consequence of the fact that it is associated with the complex realization of a denominator term.

Equations (E5.4-1a) and (E5.4-1b) show that a profile benchmark falls at $\omega \tau_i = 1.0$. At this value the magnitude and the phase of the transfer profile $T(\omega)$ will become

$$\frac{|v_i|}{|v_s|} = \frac{1}{\sqrt{1 + 1}} = \frac{1}{\sqrt{2}}$$

and

$$\phi = -\tan^{-1}(1) = -45^\circ$$

Applying this condition to the component values of the circuit figure E5.4-1(a):

$$\tau_i = R_i C_i = 16k\Omega \times 1nF = 16 \mu s$$

for which the benchmark (radian) frequency will be

$$\omega = \omega_i = 1/\tau_i = 1/16\mu s$$

This result corresponds to cyclic frequency $f_i \approx 0.16 \omega_i \approx 0.16/16\mu = .01 \text{ MHz} = 10 \text{ kHz}$

which is exactly what the pspice rendition shows (figures below)

Note that since frequency is expected to range over many orders of magnitude, the frequency scale is always a decade scale. Otherwise the profile features would become squashed left and virtually impossible to see.
There is another nomenclature teaching point defined by this exercise is associated with a different scale for the magnitude plot as represented by the figures below:

Figure 5.4-1(d) log-log magnitude plot.

Figure 5.4-1(e) dB magnitude scale with -3dB corner marked by the cursor.

Figure 5.4-1(d) is figure 5.4-1(b) with the y-axis reset as a decade scale. Since the frequency axis is a decade scale, this form of magnitude plot is now a log-log plot and the magnitude profile now appears to show a distinct roll-off ‘corner’ at \( f = f_1 \).

If asymptotes had been included, they would intersect exactly at \( f = f_1 = 10 \text{ kHz} \).

Figure 5.4-1(e) is the magnitude plot using another form of decade scale for the y-axis. In this case \(|T|\) is in dB measure, the order-of-magnitude scale for signal power. For dB measure a factor of 10 change in the amplitude ratio corresponds to 20dB change in power. So the factor of \( 2^{-1/2} \) corresponds to

\[
20 \log(1/\sqrt{2}) = -3dB
\]

Therefore at benchmark frequency \( \omega_1 \) ( \( f_i = 0.16 \omega_1 \) ) the magnitude profile rolls off by 3dB and this ‘corner’ is then denoted as the (−) 3dB corner.

Beyond the 3dB corner the profile rolls off by approximately an order of magnitude per decade. For a single time constant profile (as in this case) it is -20dB/decade (= one decade per decade). Roll-off for the frequency profile is always identified in dB/decade. Checking the axis scales for figure 5.4-1(e) a 20dB/decade roll-off should be apparent.

Example 5.4-1 has been used to make a few teaching points and as a means of identifying the nomenclature associated with frequency profiles, in particular those circuits in which the response is defined by a single time constant.
The pspice renditions also reflect the fact that magnitude and phase plots are the standard for defining the frequency profile. These usually are identified as the Bode magnitude plot and the Bode phase plot after their originator, Hendrik Bode.

Example 5.4-1 also reflects the fact that the profile has a pass-band character. The pass-band is the range of frequencies that are not suppressed by roll-off. The magnitude plots, particularly figure 5.4-1(d), show that the $|T(\omega)|$ for frequencies below $f_1$ will pass and those above will exponentially diminish by 20dB/decade. So Example 5.4-1 is a low-pass profile.

This profile might be compared to another single-time constant teaching example (Example 5.4-2) as follows:

**EXAMPLE 5.4-2:** (a) Determine the transfer function vs frequency and (b) render the circuit in pspice.

**SOLUTION:** (a) Using KVL

$$v_S = i \times (R_2 + j\omega L_1) \quad \text{and} \quad v_T = i \times j\omega L_1$$

So

$$T(\omega) = \frac{v_T}{v_S} = \frac{j\omega L_1}{R_2 + j\omega L_1} = \frac{j\omega L_1 / R_2}{1 + j\omega L_1 / R_2} = \frac{j\omega \tau_2}{1 + j\omega \tau_2} \quad \text(E5.4-2a)$$

where

$$\tau_2 = L_1 / R_2 = 160\mu\text{H}/10\Omega = 16 \mu\text{s}$$

This (benchmark) time constant corresponds to $f_2 = 0.16 \times (1/16\mu\text{s}) = .01\text{MHz} = 10\text{kHz}$

Transfer function $T$ is determined by the ratio of the magnitudes of the numerator and the denominator:

$$\left| \frac{v_T}{v_S} \right| = \frac{\omega \tau_2}{\sqrt{1 + (\omega \tau_2)^2}} \quad \text(E5.4-2b)$$

The phase shift associated with the (complex) denominator is $\phi = \tan^{-1}(\omega \tau_2)$

Since the numerator of equation (E5.4-2a) has a phase shift $\phi_N = 90^\circ$ for all frequencies then the phase shift for $T = \text{Num/Denom}$ is

$$\phi = 90^\circ - \tan^{-1}(\omega \tau_2) \quad \text(E5.4-2c)$$

(b) This is exactly what the pspice rendition of the Bode plots show (figures below/next page)
Take note that for Figure 5.4-2(b) as $\omega \to 0$ the magnitude goes to zero by an order of magnitude per decade (20dB/decade roll-off). So the lower frequencies are suppressed and this profile is then a single-time constant (STC) high-pass profile.

Otherwise these Bode plots represent the same teaching points as the previous example, namely a 3dB roll-off corner (at $f_1 = 10kHz$) and a 45° phase shift at the ‘corner frequency’.

These two teaching examples emphasize that the frequency response is affiliated with the time constants of the circuit.

Circuits with many reactive components will have an abundance of time constants and a much more complicated frequency profile, and so a concise mathematical representation of the frequency profile may be difficult to realize. It is therefore in order to clean up the process so that the generic story can be discerned, usually as approximations and as overall profiles. This charter can be accomplished in part by the substitution

$$s = j \omega$$

for which the analysis becomes more algebraic. Example 5.4-1 would then have an algebraic profile

$$T(\omega) = \frac{v_1}{v_s} = \frac{G_1}{G_1 + sC_1}$$

for which the denominator has a pole in the s-plane at $s = -G_1/C_1$

Note that this pole corresponds to the same STC benchmark frequency as identified by analysis with complex numbers, i.e.

$$\omega_1 = G_1/C_1$$
This analogy implies that a zero in the s-plane implies the existence of a characteristic frequency at that value. Assessment with complex arithmetic does not need to acknowledge the implied negative and only benchmarks these zeros in terms of their STC frequency corners. As illustrated by equation (5.4-6) a zero in the denominator is a pole that indexes the presence of a roll-off corner at that value.

Equation (5.4-4) has another implication. The use of $s = j\omega$ is the principle behind the differential equation analysis method of Laplace transforms, for which the outcome is a function $F(s)$. If the circuit is cast in terms of reactive elements with derivative response then it will be represented by a set of differential equations. Usually circuits has no need this level of mathematics overhead since $F(s)$ is very readily resolved by use of KCL and KVL.

Since most circuits can be analyzed in terms of linear components, the expectation is that the transfer function will be a ratio of two polynomials $N(s)$ and $D(s)$, according to

$$T(\omega) = \frac{N(s)}{D(s)} \quad (5.4-7)$$

The fundamental theorem of algebra says that any polynomial can be resolved into a product of zeros, i.e

$$F(s) = k(s + z_1)(s + z_2)(s + z_3)\cdots(s + z_m) \quad (5.4-8)$$

Some of the zeros may be complex, which is consistent with the fact that the s-plane is the complex plane. The transfer function given by equation (5.4-7) can then be generalized as

$$T(s) = \frac{N(s)}{D(s)} = K \frac{(s + z_1)(s + z_2)(s + z_3)\cdots(s + z_m)}{(s + p_1)(s + p_2)(s + p_3)\cdots(s + p_n)} \quad (5.4-9)$$

The notation emphasizes that the numerator terms are zeros of the transfer function and the denominator terms are poles of the transfer function. Poles cause a roll-off of $T(s)$. Zeros cause a roll-up of $T(s)$.

This fact is illustrated by the following example (Example 5.4-3).

**EXAMPLE 5.4-3:** (a) Determine the approximate transfer function $|T|$ vs frequency, (b) render the circuit in pspice and (c) Compare the two.

**SOLUTION:** (a) Use of nodal analysis at $V_3$ gives

$$v_3(G_1 + G_2 + s(C_1 + C_2)) - v_G(G_1 + sC_1) = 0$$

![Figure E5.4-3(a). Example RC circuit.](image-url)
For which \[ T(s) = \frac{v_2}{v_s} = \frac{G_1 + sC_1}{(G_1 + s)^2 + s(C_1 + C_2)} \]

Note that \( T(s) \) is of the form
\[ T(s) = \frac{k_N (s + z_1)}{k_D (s + p_1)} \]

This shows the utility of s-plane analysis. Corner frequencies can be determined by inspection, i.e.
\[ z_1 = \frac{G_1}{C_1} = \frac{1}{(50 \times 50)} = 0.4 \text{Mr/s} = 400 \text{kr/s} \quad \rightarrow \quad 0.16 \times 400k = 64 \text{kHz} \]

And \[ p_1 = \frac{(G_1 + G_2)}{(C_1 + C_2)} = \frac{0.02 + 0.005}{200 + 50} \text{Gr/s} = 0.1 \text{Mr/s} = 100 \text{kr/s} \]
\[ \rightarrow \quad 0.16 \times 100k = 16 \text{kHz} \]

Note that \( T(s) \) has both a pole and a zero, so there will be both a roll-off and a roll-up.

The magnitude character of the profile is readily determined by what happens in the limits, i.e.

if \( s \rightarrow 0 \) (same as \( \omega \rightarrow 0 \)) then
\[ |T| \rightarrow \frac{G_1}{G_1 + G_2} = \frac{0.02}{0.02 + 0.005} = 0.8 \]

And if \( s \rightarrow \infty \) (same as \( \omega \rightarrow \infty \)) then
\[ |T| \rightarrow \frac{C_1}{C_1 + C_2} = \frac{50}{50 + 200} = 0.2 \]

If these corners and levels are mapped on an ‘asymptote approximate’ Bode magnitude plot then the rough expectation of the profile will look like figure E5.4-3(b).

**Figure E5.4-3(b).** Sketch of Bode magnitude plot expectation (sharp corner approximation).

(b) The pspice rendition of the circuit and of the above sketch is shown by figure E5.4-3(c).
Example 5.4-3 has been a teaching example of a different kind. It placed an emphasis more on the magnitude profile of the frequency response and the limit values as $s \to 0$ and $s \to \infty$. These are perceptive techniques for the frequency response characteristics benchmarked by the corners and the time constants that define the corners.

The example also confirms that s-plane analysis is friendly. And this is of benefit to both the analytical assessment and the simulation assessment. The analytical assessment is interested in the characteristic frequencies and magnitude limits. The simulation assessment is oriented toward the detailed profile.

The simulation analysis has a mission that may reach into options in which components are non-linear, for which the mathematics must be iterated and circuit angels fear to tread.

### 5.5 SECOND-ORDER FREQUENCY PROFILES and RESONANT CIRCUITS

The largest category of circuit applications are vested in second-order profiles since they not only include low-pass and high-pass options, but also band-pass and other second-order options. By second-order profiles that means that the transfer profile

$$
T(s) = \frac{N(s)}{D(s)}
$$

Has a numerator $N(s)$ that is quadratic and a denominator $D(s)$ that is quadratic. Second-order profiles are therefore also called ‘bi-quadratic’ profiles.
The denominator $D(s)$ holds the key to the baseline profile characteristics since it defines the poles. It can be written as

$$D(s) = s^2 + Bs + C$$

But in order for the coefficients to be consistent with the frequency charter, $C$ must be of the form $(\text{frequency})^2$ and $B$ must be of the form of frequency. The elected coefficients are then:

$$C = \omega_0^2 \quad \text{and} \quad B = k \omega_0 = \omega_0/Q$$

for which $\omega_0$ (or $f_0$) is defined as the characteristic frequency of the quadratic profile. The denominator will then have a form consistent with its quadratic charter in the frequency domain. The standard denominator for a biquad transfer function is

$$D(s) = s^2 + \omega_0/Q + \omega_0^2$$

and the biquadratic transfer function will then be of the form

$$T(s) = \frac{as^2 + b \omega_0 s + c \omega_0^2}{s^2 + s \omega_0/Q + \omega_0^2}$$

The numerator $N(s)$ holds the key to the type of profile. For convenience and for the same reasons of coefficient choice for the denominator coefficients, $\omega_0$ is included in the numerator profile

**Case 1:** $a = b = 0$ and $c \neq 0$ then

$$T(s) = \frac{c \omega_0^2}{s^2 + s \omega_0/Q + \omega_0^2}$$

= finite when $s \to 0$ and $= 0$ when $s \to \infty$. So this profile is a low-pass profile.

**Case 2:** $a \neq 0$ and $b = c = 0$ then

$$T(s) = \frac{as^2}{s^2 + s \omega_0/Q + \omega_0^2}$$

= zero when $s \to 0$ and finite when $s \to \infty$. So this profile is a high-pass (HP) profile.

**Case 3:** $b \neq 0$ and $a = c = 0$ then

$$T(s) = \frac{bs}{s^2 + s \omega_0/Q + \omega_0^2}$$

= zero when $s \to 0$ and zero when $s \to \infty$. But it is finite when $s = j \omega_0$. So this type of profile is a band-pass (BP) profile.

The biquadratic form can also have other type profiles. If the numerator is of the form
\[ N(s) = K \left( s^2 + sk\omega_0 + \omega_0^2 \right) \]

The profile will be a band-stop (BS) profile. If \( k \) happens to be 0 then the profile will go to zero at \( s = j\omega_0 \) and completely stop (or block) frequency \( \omega_0 \).

If the numerator is
\[ N(s) = K \left( s^2 - s\omega_0/Q + \omega_0^2 \right) \]

The profile is an all-pass profile and does nothing except shift the frequency. So it is also called a phase-shifter circuit.

The normalized form of the bandpass (BP) profile is cited as a benchmark of the biquadratic profiles because it is symmetric about \( \omega = \omega_0 \). It will have \(|T(\omega = \omega_0)| = 1\) if normalized by use of \( b = \omega_0/Q \).

This is the profile represented by figure 5.5-1. For this rendition (pspice) the center (= characteristic frequency is \( f_0 = 1 \text{MHz} \) and \( Q = 5 \). The 3db ( = \( 1/\sqrt{2} \) ) levels are marked by the two cursors.

![Figure 5.5-1. Normalized bandpass biquad profile.](image)

The mathematical significance of the two cursors is that they relate to the level condition

\[ |T(s)| = \left| \frac{s\omega_0/Q}{s^2 + s\omega_0/Q + \omega_0^2} \right| = \frac{1}{\sqrt{2}} \quad \text{(5.5-7)} \]

Since \( s = j\omega \) then the magnitude of the denominator will be

\[ |D(s)| = \left| (j\omega)^2 + j\omega(\omega_0/Q) + \omega_0^2 \right| = \left| \omega_0^2 - \omega^2 + j\omega(\omega_0/Q) \right| = \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2(\omega_0/Q)^2} \]
So rationalizing the denominator in equation (5.5-7) and clearing fractions gives

$$2\omega^2 (\omega_0/Q) = (\omega_0^2 - \omega^2)^2 + \omega^2 (\omega_0/Q)^2$$  \hspace{1cm} (5.5-8)

which can be rewritten as

$$(\omega_0^2 - \omega^2)^2 = \omega^2 (\omega_0/Q)^2$$  \hspace{1cm} (5.5-9)

Taking square roots

$$(\omega_0^2 - \omega^2) = \pm \omega (\omega_0/Q)$$ \hspace{1cm} or \hspace{1cm} $$\omega^2 \pm \omega (\omega_0/Q) - \omega_0^2 = 0$$

which is either two quadratic equations or a quartic equation. As a quartic there should be four roots = values of $\omega$ that fit equation (5.5-7). They are

$$\omega = \pm \frac{\omega_0}{2Q} \pm \omega_0 \sqrt{1 + \frac{1}{4Q^2}}$$  \hspace{1cm} (5.5-10)

Only two of these roots are positive and they are

$$\omega_1 = \frac{\omega_0}{2Q} + \omega_0 \sqrt{1 + \frac{1}{4Q^2}} \hspace{1cm} \omega_2 = -\frac{\omega_0}{2Q} + \omega_0 \sqrt{1 + \frac{1}{4Q^2}}$$

and this gives a relationship for the width of the bandpass profile at the 3dB level (1/sqrt(2) level of

$$\Delta \omega = \omega_1 - \omega_2 = + \frac{\omega_0}{2Q} - \left( - \frac{\omega_0}{2Q} \right) = \frac{\omega_0}{Q}$$  \hspace{1cm} (5.5-11)

Equation (5.5-11) identifies the rationale for the choice of parameter $Q$ in equation (5.5-2) since it gives

$$Q = \frac{\omega_0}{\Delta \omega} = \frac{f_0}{\Delta f}$$  \hspace{1cm} (5.5-12)

Equation (5.5-12) implies that the bandwidth $= \Delta f$ (defined at the 3dB level) of the profile is defined by the parameter $Q$. The parameter $Q$ therefore is then denoted as ‘quality factor’ for the (bandpass) profile.

For figure 5.5-1 the quality factor was set to $Q = 5.0$. This value is confirmed by the cursors inasmuch as they show a $\Delta f = 0.2\text{MHz}$ for the peak frequency $f_0 = 1.0\text{MHz}$. 

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The bandpass circuit is a representation of a resonance peak. The resonance peak will also appear with the LP and HP profiles. The parameter $Q$ is equally of significance to these profiles, whether normalized or not. For the lowpass profile (equation (5.5-4)) and its magnitudes for $s \to 0$ and $s = j\omega_0$:

$$|T(s \to 0)| = \left| \frac{c}{s^2 + s\omega_0/Q + \omega_0^2} \right|_{s \to 0} = \frac{c}{\omega_0^2}$$  \hspace{1cm} (5.5-13)

$$|T(s = j\omega_0)| = \left| \frac{c}{s^2 + s\omega_0/Q + \omega_0^2} \right|_{s \to j\omega_0} = \frac{c}{j\omega_0 \times \omega_0/Q} = \frac{cQ}{\omega_0^2}$$ \hspace{1cm} (5.5-14)

And so the ratio

$$\frac{|T(\omega_0)|}{|T(0)|} = \frac{cQ/\omega_0^2}{c/\omega_0^2} = Q$$  \hspace{1cm} (5.5-15)

which shows that the LP profile has a resonance gain of $Q$ at the characteristic frequency. This fact is confirmed by figure 5.5-2 which is a pspice rendition of an LP circuit with $Q = 5$. The cursors confirm the factor 5.0.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lowpass_biquad_profile.png}
\caption{Lowpass biquad profile.}
\end{figure}

A summary table of the (normalized) biquadratic function is given by table 5.5-1
<table>
<thead>
<tr>
<th>Type profile</th>
<th>Biquadratic form</th>
<th>Pspice rendition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandpass</td>
<td>$T(s) = \frac{s \omega_0/Q}{s^2 + s \omega_0/Q + \omega_0^2}$</td>
<td>![Bandpass Graph]</td>
</tr>
<tr>
<td>Lowpass</td>
<td>$T(s) = \frac{\omega_0^2}{s^2 + s \omega_0/Q + \omega_0^2}$</td>
<td>![Lowpass Graph]</td>
</tr>
<tr>
<td>Highpass</td>
<td>$T(s) = \frac{s^2}{s^2 + s \omega_0/Q + \omega_0^2}$</td>
<td>![Highpass Graph]</td>
</tr>
<tr>
<td>Bandstop</td>
<td>$T(s) = \frac{s^2 + \omega_0^2}{s^2 + s \omega_0/Q + \omega_0^2}$</td>
<td>![Bandstop Graph]</td>
</tr>
<tr>
<td>All-pass</td>
<td>$T(s) = \frac{s^2 - s \omega_0/Q + \omega_0^2}{s^2 + s \omega_0/Q + \omega_0^2}$</td>
<td>![All-pass Graph]</td>
</tr>
</tbody>
</table>

Table 5.5-1. Biquadratic functions
The low-pass (and high-pass) profiles also have some special-case values of $Q$ indicated by figure 5.5-3, which shows the profile with several select values for $Q$

\[ D(s) = \left[ s^2 + s \omega_0 \right]_{Q=0.5} = s^2 + 2 \omega_0 s + \omega_0^2 = (s + \omega_0)^2 \]  

Figure 5.5-3(a). Lowpass biquad profile with $Q = 0.5, 1/\sqrt{2}, 2$, and 5.

Figure 5.5-3(b). Same as 5.5-3(a) but on dB scale. Note that roll-off = -40dB/decade.

Figure 5.5-3(a) shows that the resonance peak will shrink as $Q$ is reduced. And it will be lost entirely for $Q < 1/\sqrt{2}$. The value of $Q = 1/\sqrt{2}$ is a special case corresponding to the low-pass profile being maximally flat.

Figure 5.5-3(b) shows that the roll-off for $f >> f_0$ is now -40 dB/decade, with some emphasis on the maximally flat case ($Q = 1/\sqrt{2}$). This roll-off is twice that observed for the single time constant (1st-order) profile and identifies that the order of the profile defines its roll-off (= ability to discriminate) by the rule

\[ \text{Rolloff} = 20\text{dB/dec} \times \text{(order of profile)} \]  

(5.5-16)

The other special case occurs for $Q = 0.5$ in which case the quadratic denominator becomes

\[ D(s) = \left[ s^2 + s \omega_0 \right]_{Q=0.5} = s^2 + 2 \omega_0 s + \omega_0^2 = (s + \omega_0)^2 \]  

For which both of the roots of the denominator are now real, with $s = \omega_0$.

The roots of the quadratic denominator are not normally real but complex, of the form $s = \alpha \pm j\beta$. Then:

\[ D(s) = \left[ s + (\alpha + j\beta) \right] \left[ s + (\alpha - j\beta) \right] = s^2 + 2\alpha s + (\alpha^2 + \beta^2) \]

Comparison to equation (5.5-2) identifies the relationship of $\alpha$ and $\beta$ to $Q$ and $\omega_0$. 

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(\alpha^2 + \beta^2) = \omega_0^2 \quad \quad \quad 2\alpha = \omega_0/Q

or \quad \alpha = \omega_0/2Q \quad \beta = \omega_0 \sqrt{1 - (1/2Q)^2} \quad (5.5-18)

If \( Q < 0.5 \) then \( \beta = \) real value and the roots are real. And the profile is a single-time constant form with shifted 3dB corner and -40dB/decade roll-off.

This form is sometimes written as

\[ \beta = \omega_0 \sqrt{1 - \xi^2} \quad (5.5-19) \]

Where \( \xi \equiv \) normalized damping coefficient and relates to \( Q \) as

\[ \xi = 1/2Q \quad (5.5-20) \]

And since \( 2\alpha = \omega_0/Q = 2\omega_0\xi \) then

\[ \xi = \alpha/\omega_0 \quad (5.5-21) \]

where \( \alpha \) is the signal damping coefficient.

### 5.6 RLC CIRCUITS

A common instance of a resonant circuit and biquadratic response is an RLC circuit. It incorporates the two energy storage energy components, \( L \) and \( C \). These components are complementary and their interaction thereto induces an energy resonance.

A typical \( LC \) resonant circuit might be that of the series RLC form as represented by figure 5.6-1.

![Figure 5.6-1. Series RLC circuit (with output across C).](image)

The circuit can be directly analyzed means of KVL and the impedances for \( L \) and \( C \) in the s-plane.
Equation (5.6-1) can be reduced to a form consistent with the biquadratic form of equation (5.5-3) by

\[
\frac{v_c}{v_s} = \frac{1/sC}{R + sL + 1/sC} = \frac{1/LC}{s^2 + sR/L + 1/LC}
\]

Comparisons of coefficients for the quadratic denominators gives

\[
\omega_0 = \frac{1}{\sqrt{LC}}
\]

and

\[
\omega_0/Q = R/L
\]

If equation (5.6-3) is applied to (5.6-4) and solved for \(Q\) then

\[
Q = \frac{\omega_0 L}{R} = \frac{\sqrt{L}}{\sqrt{C}}/R
\]

It is convenient and appropriate to identify

\[
R_0 = \frac{\sqrt{L}}{\sqrt{C}}
\]

as the characteristic resistance of the \(RLC\) circuit. In which case

\[
Q = \frac{R_0}{R}
\]

The inductance and the capacitance are complimentary energy storage elements. When an inductance and a capacitance share a circuit topology the capacitance will see the inductance as a short circuit and the inductance will see the capacitance as an open circuit. The action of the capacitance will be to discharge through the short circuit (the inductance) thereby effecting a current through the inductance. The action of the inductance will be to sustain current as long as possible which will cause a charge of the capacitance. The cycle continually repeats. As a consequence, charge and current bounce back and forth between the \(L\) and the \(C\) with an energy oscillation of oscillation frequency predicated by the \(L\) and \(C\) values, as reflected by equation (5.6-3). The frequency \(\omega_0\) (or \(f_0 = \omega_0/2\pi\)) is therefore a characteristic resonance due to the energy oscillation between complementary storage elements.
Ideally the \( LC \) resonance would continue forever, and can do so if there is no dissipative element. Otherwise the energy will dissipate in the resistance. The lossy effect is relative to the characteristic resistance \( R_0 \) of the \( LC \) pair as defined by equation (5.6-6).

Since all circuits include resistances (parasitic and/or leakage) the characteristic resistance \( R_0 \) gives a benchmark on which succinctly defines the profile characteristics of the (RLC) circuit topologies.

**Series RLC circuit.** With components in series the effect of the \( LC \) resonance across each of the components can be selectively assessed. The output across the capacitance was evaluated by equation (5.6-2) for which

\[
\frac{v_C}{v_S} = \frac{1/LC}{s^2 + sR/L + 1/LC} = \frac{1/LC}{s^2 + sR/L + 1/LC} = \frac{\omega_0^2}{s^2 + s\omega_0/Q + \omega_0^2}
\]

(5.6-8)

and is characteristic of a low-pass response.

On the other hand if the output were taken across the inductance then the circuit will be of the form shown by figure 5.6-2. And by KVL will have response

\[
T(s) = \frac{v_L}{v_S} = \frac{sL}{R + sL + 1/sC}
\]

which, if multiplied by \( s/L \), gives

\[
T(s) = \frac{v_L}{v_S} = \frac{sL}{R + sL + 1/sC} \times \frac{s/L}{s/L} = \frac{s^2}{s^2 + sR/L + 1/LC}
\]

This result has the same denominator as equation (5.6.1), but numerator \( s^2 \). So the output across the inductance is of the form

\[
\frac{v_L}{v_S} = \frac{sL}{R + sL + 1/sC} = \frac{\omega_0^2}{s^2 + s\omega_0/Q + \omega_0^2}
\]

(5.6-9)

which is exactly that of a high-pass form.
If the output is taken across the resistance as shown by figure 5.6-3

\[ \frac{V_R}{V_{S}} = \frac{R}{R + sL + \frac{1}{sC}} \frac{s/L}{s/L} = \frac{s\omega_0/Q}{s^2 + s\omega_0/Q + \omega_0^2} \]  

(5.6-10)

which is of the **band-pass** form.

And if the output is taken across the *LC* series pair

\[ \frac{V_{LC}}{V_{S}} = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{s^2 + \omega_0^2}{s^2 + s\omega_0/Q + \omega_0^2} \]  

(5.6-11)

which is of the **band-stop** form.

---

**EXAMPLE 5.6-1:** For the following values of *R*, *L*, and *C* in a series topology determine (a) the resonant frequency \( f_0 \) (b) the characteristic resistance \( R_0 \), (c) the quality factor \( Q \).

1. \( R = 50\, \Omega \), \( C = 40\, \text{pF} \), \( L = 10\, \mu\text{H} \)
2. \( R = 5\, \Omega \), \( C = 0.2\, \text{\mu F} \), \( L = 80\, \mu\text{H} \)

**SOLUTION:**

1. For the prefixes shown \( \frac{1}{\sqrt{\mu \times p}} = G(=10^9) \) and \( \sqrt{\mu \over p} = k(=10^3) \)

therefore \( \omega_0 = \left(\frac{1}{\sqrt{400}}\right)\text{Gr/s} = 0.5\text{Gr/s} = 50\text{Mr/s} \rightarrow f_0 = 0.16\omega_0 = 0.16 \times 50 = 8\text{MHz} \)

\( R_0 = \frac{10}{\sqrt{40}} = 0.5\, \text{k}\Omega = \textbf{500}\, \Omega \)  

\( Q = \frac{R_0}{R} = \frac{500}{50} = \textbf{10} \)
(2) For the prefixes shown \[
\frac{1}{\sqrt{\mu \times \mu}} = M (= 10^9) \quad \text{and} \quad \sqrt{\frac{\mu}{\mu}} = 1
\]
therefore \[
\omega_0 = \left(\frac{1}{\sqrt{0.2 \times 80}}\right) \text{Mr/s} = 0.25 \text{Mrs} = 250 \text{kr/s} \quad \rightarrow \quad f_0 = \frac{0.16 \omega_0}{2.5} = 0.16 \times 250 = 40 \text{kHz}
\]
\[
R_0 = \frac{80}{\sqrt{0.2}} = 20 \Omega
\]
\[
Q = \frac{R_0}{R} = \frac{20}{5} = 5.0
\]
Mastery of the prefixes lets the RLC analysis to be done virtually by inspection!

Component measures on the order of pF and \(\mu\)H correspond to MHz. Component measure on the order of unit pF and 10nH correspond to UHF frequencies.

Large inductances and small series resistances give a higher \(Q\).

**EXAMPLE 5.6-2:** For a series topology of parasitics for which \(R = 2.5 \Omega, L = 10\text{nH}, \text{and } C = 4\text{pF}, \) determine \(f_0, R_0, \text{and } Q\).

**SOLUTION:** For the prefixes shown \[
\frac{1}{\sqrt{\mu \times \mu}} = G (= 10^9) \quad \text{and} \quad \sqrt{\frac{\mu}{\mu}} = k (= 10^3)
\]
therefore \[
\omega_0 = \left(\frac{1}{\sqrt{0.1 \times 4}}\right) \text{Gr/s} = 5.0 \text{Gr/s} \quad \rightarrow \quad f_0 = \frac{0.16 \omega_0}{2.5} = 0.16 \times 5.0 = 800 \text{MHz}
\]
\[
R_0 = \frac{0.1}{\sqrt{4}} = 0.05 \text{k\Omega} = 50 \Omega
\]
\[
Q = \frac{R_0}{R} = \frac{50}{2.5} = 20
\]
*The secret to quick analysis is in the prefixes.

Also take note that for the RLC series circuit the type of profile (LP, BP, HP, BS) will depend on the type component (or components) across which output is taken.
Parallel RLC circuit. The parallel RLC circuit is a fundamentally different topology from the series RLC, but also has resonance between $L$ and $C$ will occur. It is represented below.

If nodal analysis is applied at $V_P$ then

$$v_P(G + sC + 1/sL) - v_S G = 0$$

which results in transfer function

$$\frac{v_P}{v_S} = \frac{G}{G + sC + 1/sL} \quad (5.6-12)$$

If numerator and denominator are multiplied by $s/C$ then

$$\frac{v_P}{v_S} = \frac{G}{G + sC + 1/sL} \times \frac{s/C}{s/C} = \frac{sG/C}{s^2 + sG/C + 1/LC}$$

which simplifies to

$$\frac{v_P}{v_S} = \frac{s \omega_0 / Q}{s^2 + s \omega_0 / Q + \omega_0^2} \quad (5.6-13)$$

and is of the bandpass form.

The term $G/C = \omega_0 / Q$ identifies the specifics of the component relationship to $Q$

$$Q = \omega_0 \times C / G = R \left/ \frac{\sqrt{L}}{\sqrt{C}} \right. = R/R_0 \quad (5.6-14)$$

Note that this result is the reciprocal of that of series RLC topology.
Inductance, however, is never free of resistance. The material in the conduction loop will invariably include a finite resistance in series with $L$, a modification of figure 5.6-4 as represented by figure 5.6-5.

![Figure 5.6-5. Parallel RLC topology including finite series resistance.](image)

If a nodal analysis is executed at $V_O$ then

$$v_O [G + sC + 1/(R_S + sL)] - G v_S = 0$$

After some fun algebra this gives transfer function

$$\frac{v_O}{v_S} = \frac{G(R_S + sL)/LC}{s^2 + s(R_S/L + G/C) + (1 + GR_S)/LC}$$  \hspace{1cm} (5.6-15)

This equation gives a shifted form of resonance frequency $\omega_0$

$$\omega'_0 = \sqrt{1 + R_S/R} \times \omega_0$$  \hspace{1cm} (5.6-16)

and a value of $Q$ (via more fun algebra) of

$$Q = \frac{\sqrt{1 + R_S/R}}{R_S/R_O + R_O/R}$$  \hspace{1cm} (5.6-17)

In the limit $R \to $ large $\omega'_0 \to \omega_0$ and

$$Q \to \sqrt{1 + 0/(R_S/R_O + 0)} = R_o/R_S$$  \hspace{1cm} (5.6-18)

Equations (5.6-17) and (5.6-18) imply that the ultimate limit in $Q$ is the series resistance $R_S$ to $L$, which is most probably the resistance of the wire that forms the inductive coil.

The $RLC$ series sub-circuit in figure 5.6-5 then is the key to the response limits of the $RLC$ resonance. If equation (5.6-5) is applied to the above analysis then the bandwidth limit will be

$$\Delta f = \frac{1}{2\pi} \frac{R_S}{L}$$  \hspace{1cm} (5.6-19)
consistent with equations (5.6-4) and (5.5-12).

For all RLC circuits the frequency domain characteristics of $f_0$ and $Q$ also define the impulse (bell ‘tap’) of the circuit. If an impulse is applied to a resonant circuit it will ‘ring’ (like a bell) approximately at frequency $f_0$ with amplitude decay defined by the normalized damping factor $\xi$.

A pspice rendition of a series RLC circuit and its ring-down response is shown below. Characteristics are elected to be $f_0 = 1\text{MHz}$ with quality factor $Q = 5$. Component values are parameterized according to equations (5.6-3), (5.6-6), and (5.6-7).

The ring-down response is useful for high sensitivity measurements and detection of minute levels of contamination.

**EXAMPLE 5.6-3:** Execute the following circuit in pspice and (1) show a Bode magnitude plot and (2) an impulse response.

From the Bode plot (a) extract $f_0$ and $Q$ and (b) confirm these values by extracting the ringing frequency and the damping factor $\xi$ from the impulse response.

**SOLUTION:** The Bode Magnitude plot is shown by figure E5.6-3(b)

This is a low-pass form, consistent with the output across capacitance $C$.

From the figure and cursor values:

$$f_0 = 1.0\text{MHz} \text{ (by inspection)}$$

$$Q = \frac{|T(f = 1\text{MHz})|}{|T(f = 0)|} = \frac{5.0246}{1.010} \approx 5.0$$

**Figure E5.6-3(a).** RLC series example. Bode magnitude plot.
For the pulse response the cursors identify the essential information. The difference between cursors on the time scale gives

\[ \Delta t \approx 1.0 \mu s \]

So the cyclic frequency is

\[ f_0 = \frac{1}{\Delta t} \approx 1/1.0 \mu s = 1 \text{MHz} \]

The two amplitudes marked by the cursors are

\[ V(t_1) = 1.7269 \quad \text{and} \quad V(t_2) = 1.3897 \]

These values represent the ‘ring-down’ generated by the 1.0V pulse. So the ringing amplitudes are

\[ \Delta V(t_1) = 1.7269 - 1.0 = 0.7269 \quad \text{and} \quad \Delta V(t_2) = 1.3897 - 1.0 = 0.3897 \]

which must obey

\[ \Delta V(t) = \Delta V(0)e^{-\sigma t} \]

with ratio between any two amplitudes

\[ \frac{\Delta V(t_2)}{\Delta V(t_1)} = \frac{\Delta V(0)e^{-\sigma t_2}}{\Delta V(0)e^{-\sigma t_1}} = e^{-\sigma(t_2-t_1)} = e^{-\sigma \Delta t} \]

Taking the inverse

\[ \ln \left( \frac{\Delta V(t_2)}{\Delta V(t_1)} \right) = -\sigma \times \Delta t = -\sigma \times \frac{1}{f_0} = -\omega_0 \xi \times \frac{1}{f_0} = -2\pi \xi \]

for which

\[ \ln \left( \frac{0.3897}{0.7269} \right) = -0.623 = -2\pi \xi \]

and

\[ \xi \approx \frac{1}{2\pi} \times 0.623 = 0.1 \]

\[ = \text{ same as } 1/2Q = 1/(2 \times 5) \]

Compare the above example to example 4.3-1.
PORTFOLIO and SUMMARY

Sinusoidal analysis:

Complex number form \( x(t) = x_0 e^{j\omega t} \),

where \( x(t) = V(t) \) or \( I(t) \)

impedance \( Z = R + jX = |Z| e^{j\phi} \)

where \( X = \text{reactance} \) \( |Z| = \sqrt{R^2 + X^2} \) \( \phi = \tan^{-1}(X/R) \)

admittance \( Y = G + jB = |Y| e^{j\phi} \)

where \( B = \text{susceptance} \) \( |Y| = \sqrt{G^2 + B^2} \) \( \phi = \tan^{-1}(B/G) \)

Modified Ohm’s law \( i = Yv \) \( v = Z_i \) and \( Y = 1/Z \)

Kirchoff laws: \( KCL: \sum_{\text{node}} i_K = 0 \) \( KVL: \sum_{\text{loop}} v_K = 0 \)

Inductance: \( Z_L = j\omega L = sL \) \( C = j\omega C = sC \)

Single-time constant profiles (first-order profiles): \( \tau_c = C/G = RC \) \( \tau_L = L/R \)

(1) Low-pass

\[ \left| \frac{v_1}{v_s} \right| = \frac{1}{\sqrt{1 + (\omega \tau_1)^2}} \]

At \( \omega = \omega_1 = 1/\tau_1 \) = 3dB corner

For which \( \frac{v_1}{v_s} = \frac{1}{\sqrt{2}} \) with roll-off = -20dB/decade

and \( \phi = -\tan^{-1}(1.0) = -45^\circ \)

(2) High-pass

\[ \left| \frac{v_1}{v_s} \right| = \frac{\omega \tau_2}{\sqrt{1 + (\omega \tau_2)^2}} \]

At \( \omega = \omega_2 = 1/\tau_2 \) = 3dB corner

For which \( \frac{v_1}{v_s} = \frac{1}{\sqrt{2}} \) with roll-off = -20dB/decade

and \( \phi = 90^\circ - \tan^{-1}(1.0) = +45^\circ \)
Biquadratic profiles

\[ T(s) = \frac{as^2 + b\omega_0 s + c\omega_0^2}{s^2 + s\omega_0/Q + \omega_0^2} \]
\[ \omega_0 = \text{characteristic frequency} = 2\pi f_0 \quad Q = \text{quality factor} \]

\[ D(s) \text{ has roots } s = -\alpha \pm j\beta \quad \alpha = \omega_0 / 2Q \quad \beta = \omega_0 \sqrt{1 - (1/2Q)^2} = \omega_0 \sqrt{1 - \xi^2} \]

where \( \xi = \text{normalized damping coefficient} \)
\[ \xi = 1/2Q \quad \xi = \alpha / \omega_0 \]

<table>
<thead>
<tr>
<th>Table 5.5-1. Biquadratic functions</th>
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<tbody>
<tr>
<td>Type profile</td>
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<tr>
<td>----------------</td>
</tr>
<tr>
<td>Bandpass</td>
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<tr>
<td>Lowpass</td>
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<tr>
<td>Highpass</td>
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<tr>
<td>Bandstop</td>
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<tr>
<td>All-pass (phase shifter)</td>
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</tbody>
</table>
RLC circuits: (Biquadratic forms)

characteristic frequency = \( \omega_0 = \frac{1}{\sqrt{LC}} \)
characteristic resistance = \( R_0 = \sqrt{\frac{L}{C}} \)

(1) Series RLC

Output across \( C \): = low-pass with gain peaking \( \frac{|F(\omega_0)|}{|F(0)|} = Q \)
Output across \( L \): = high-pass with gain peaking \( \frac{|F(\omega_0)|}{|F(\infty)|} = Q \)

Output across \( R \): = band-pass with bandwidth \( \Delta f = f_0/Q \)
\( \Delta f = \frac{1}{2\pi} \frac{R}{L} \)
Output across \( LC \): = band-stop

(2) Parallel RLC

Output across \( R \): = band-pass with bandwidth \( \Delta f = f_0/Q \)
\( \Delta f = \frac{1}{2\pi} \frac{C}{R} \)

If a finite series (parasitic) resistance is included with \( L \) then

\( Q = \frac{R}{R_0} \)
\( Q = \sqrt{1 + \frac{R_s}{R} / \frac{R_s}{R_0} + \frac{R_0}{R}} \)

So in the limit of \( R \to \) large \( Q \to \frac{R_0}{R_s} \) (series \( Q \))

Shortcuts:

\( L = 16\mu H, \ C = 400pF \) \( \to \) \( f_0 = 2.0 \) MHz \( R_0 = 200\Omega \) \( (= RF) \)
\( L = 10nH, \ C = 4pF \) \( \to \) \( f_0 = 800 \) MHz \( R_0 = 50\Omega \) \( (= UHF) \)

Impulse transients and frequency response

\( Q = \frac{|F(\omega_0)|}{|F(0)|} \)
\( \xi \approx \frac{1}{2\pi} \ln \left( \frac{\Delta V(t_2)}{\Delta V(t_1)} \right) \)
\( \xi = 1/2Q \)