CHAPTER 7. MAGNETIC CIRCUITS

7.1 MAGNETIC FIELDS

Since the magnetic field exists only in dipole or multipole form, it is not as readily contained as the electric field. It is a close cousin to the electric field in that it is represented by flux (field) lines. Because this context of flux lines is the identifying feature of the magnetic field, it is also denoted as the magnetic flux density, with measure of Webers/m². The Weber is a somewhat nebulous unit, but does give a means to quantify the stored energy, as has been represented by the inductance component and its charge/discharge behavior in circuits. The magnetic field is also denoted as the B-field, and relates to the magnetic permeability of materials since more permeable materials accommodate and concentrate more flux. Although Webers are not as quantifiable as are Coulombs as an energy-bearing measure, a crowding of flux lines (greater density of Webers) is assertively equivalent to a stronger B-field, just like a crowding of charge (greater density of Coulombs) is assertively equivalent to a stronger E-field density.

Collateral to the magnetic flux density and related by a constant of proportionality is the magnetic field intensity, also known as the H-field. It is a magnetic field generated by a current loop. It has a close analogy to the electric field in that it exists externally to permeable materials. Both the E-field and the H-field are generated fields. The E-field is a consequence of voltage across a gap or structure. The H-field is generated by current in a loop or multiple loop s. The two types of fields are consistent and complementary. And in their roles of energy storage form the circuit components of capacitance and inductance.

The field relationships for (1) the electric field and (2) the magnetic field are

\[ \vec{D} = \varepsilon \vec{E} \]
\[ \vec{B} = \mu \vec{H} \]

and are taken directly from chapter 3. In correlation of field nomenclature the E-field is in Volt/m and the H-field is in Amp/m (or Amp-turns/m). And although it does not play as much of a role in circuit definitions as does the B-field (in Wb/m²), the D-field (displacement field) is in Coul/m².

In analogy to the electric circuit and its charge dynamics within a conductive topology, a magnetic circuit is a topology with flux dynamics within a permeable topology. The two topologies are interrelated and interconnected. Electrical current defines magnetic flux and electrical current is a flow of charge through a conduction topology. Chapter 3 defines magnetic flux and electrical current is a flow of charge through a conduction topology. It did not address permeable topologies and the option of flux linkages in the magnetic field domain.
7.2 RELUCTANCE and INDUCTANCE

Permeable materials transport magnetic flux. The containment and transport path for magnetic flux is a construct like that shown by the figure below, a repeat from Chapter 3. It is characterized by a measure defined as reluctance $R$.

![Figure 3.3-1. Simplest geometrical form of magnetic energy storage component.](image)

Reluctance is the magnetic field equivalent to resistance $R$. Resistance is the containment and transport construct for electrical charge. The concepts are completely analogous. And therefore so is the mathematics.

The loop of permeable material as defined by reluctance $R$ confines and guides magnetic flux $\phi$ generated by the electromagnetic coil with strength $NI$. The cause-and-effect relationship is then

$$R = \frac{NI}{\phi} = \frac{NI}{BA} = \frac{1}{\mu A}$$  \hspace{1cm} (3.3-6)

Equation (3.3-6) unmistakably resembles the equation (2.1-2b) for resistance. But magnetic circuits are also a subset of electric circuits in the form of inductance, which is an energy-storage measure. So it is essential that this context be included with the geometrical and reluctance relationship as identified by equation (3.3-8).

$$L = \frac{N^2}{R} = \frac{N^2}{1/\mu} = N^2 \times \frac{A}{\ell}$$  \hspace{1cm} (3.3-8)

The fact that change in the flux induces a voltage in the set of loops (and is the reason for the $N^2$) becomes significant since voltage will then be induced in any conductance loop in the vicinity of the change in magnetic flux, particularly the flux that is transported elsewhere by the magnetic circuit. The inductance itself is an example of how the energy contained in the reluctance loop is both applied and extracted, which can be done with a multiplicity defined by the loop count $N$.

Consider the magnetic circuit defined by the construct of example 7.2-1.
EXAMPLE 7.2-1: Determine (a) the reluctance and (b) the inductance for each of the coils assuming that the core is soft iron with $\mu_r = 2000$.

SOLUTION:

For $\mu_0 = 4\pi \text{nH/cm}$

The reluctances of the core and gap are

$$R_{\text{Core}} = \frac{1}{\mu_r \mu_0} \frac{\ell}{A} \approx \frac{1}{4\pi} \times \frac{1}{2000} \times \frac{24}{2^2} \approx 0.08 \times \frac{6}{2000} = 0.00024 \, \text{nH}^{-1} = 0.24 \, \text{MA/Wb}$$

(since 1 Henry = 1Weber/Ampere)

$$R_{\text{Gap}} = \frac{1}{\mu_r \mu_0} \frac{\ell}{A} \approx \frac{1}{4\pi} \times \frac{1}{1.0} \times \frac{0.4}{2^2} \approx 0.08 \times 0.1 = 0.008 \, \text{nH}^{-1} = 8 \, \text{MA/Wb}$$

And since the reluctances are in series the reluctance of the magnetic core topology shown is

$$R = R_{\text{Core}} + R_{\text{Gap}} = 0.24 + 8.0 \approx 8.24 \, \text{MA/Wb}$$

where the less permeable gap dominates the result. (Think about series resistances. The less conductive part will have a much higher resistance than the others in the series string and will dominate)

The inductances are then

$$L_1 = \frac{N_1^2}{R} = \frac{10^2}{8.24} \approx 12 \, \mu\text{H}$$

$$L_2 = \frac{N_2^2}{R} = \frac{20^2}{8.24} \approx 48 \, \mu\text{H}$$

Note that the inductances in the above example do NOT add when they are wired in series. When wired in series the flux from one links to that of the other creating an inductance on the order of

$$L_{1+2} = \frac{30^2}{8.24} \approx 108 \, \mu\text{H}.$$
The flux linkage aspect of magnetic circuits is an important aspect of magnetic circuits. The use of a defined reluctance shared by two separated windings (inductances) allows their fluxes $\phi_1$ and $\phi_2$ to be shared, for which signal energy can then pass from one inductance to the other.

In circuits that have more than one inductance on the board, a spurious flux link can occur and unintentionally pass a signal from one part of a circuit to another part. This phenomenon is called ‘cross-talk’ and is usually not welcome.

### 7.3 FLUX-LINKED INDUCTANCES AND THE IDEAL TRANSFORMER

The fact that permeable materials enable magnetic flux to be contained within its separated path is a significant added value to electronic circuits. The magnetic flux path does two things in behalf of the electronics circuit: (1) coupling isolation and (2) scaled transfer of electrical signal energy. Depending on the relaxation constraints levied by the permeable materials it may have some frequency limitations, but otherwise the principle is straightforward. The basic principle of flux-linked inductances is represented by the two-winding permeable ring of figure 7.3-1.

![Figure 7.3-1. Use of reluctance path for flux linkage.](image)

Figure 7.3-1 is not unlike other two-port formulations other than the fact that it employs magnetic flux instead of an electrical network. Each set of current loops will create a separate flux $\phi$ as defined by equation (3.3-6), restated (for emphasis) by equation (7.3-1).

$$\phi = NI / R$$  \hspace{1cm} (7.3-1)

Since both windings have the same reluctance path the fluxes which are generated will add. And the flux generated by one will be seen by the other.
The relationship for induced voltage is defined by Faraday’s law. Faraday’s law states that the induced voltage is proportional to the change of flux with respect to time. If there is a winding of $N$ links (loops), they add in series, giving a multiplying factor of $N$ and the relationship becomes

$$v = N \frac{d\phi}{dt} \quad (7.3-2)$$

If two windings are contributing flux, then $\phi = \phi_1 + \phi_2$ and voltage $v_1$ will then be

$$v_1 = N_1 \frac{d}{dt}(\phi_1 + \phi_2) = N_1 \frac{d\phi_1}{dt} + N_1 \frac{d\phi_2}{dt}$$

By equation (3.3-6) the flux generated by a winding is $\phi = NI/R$, so that

$$v_1 = N_1 \frac{d}{dt}(N_1I_1/R) + N_1 \frac{d}{dt}(N_2I_2/R) = (N_1^2/R) \frac{dI_1}{dt} + (N_1N_2/R) \frac{dI_2}{dt}$$

$$= L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} \quad (7.3-3a)$$

Similarly

$$v_2 = N_2 \frac{d}{dt}(N_1I_1/R) + N_2 \frac{d}{dt}(N_2I_2/R) = (N_2N_1/R) \frac{dI_1}{dt} + (N_2^2/R) \frac{dI_2}{dt}$$

$$= M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} \quad (7.3-3b)$$

for which the two inductances $L_1$ and $L_2$ now are cross-coupled by a mutual inductance $M$

$$M = N_1N_2/R \quad = N_1N_2/R \quad (7.3-4)$$

The order of the windings $N_1$ and $N_2$ does not matter. As a matter of nomenclature, $L_1$ and $L_2$ are identified as self-inductances, consistent with the definition given by equation (3.3-8). The set of inductance terms are then

$$L_1 = N_1^2/R \quad (7.3-5a)$$
$$L_2 = N_2^2/R \quad (7.3-5b)$$
$$M = \sqrt{L_1L_2} \quad (7.3-5c)$$

Equation (7.3-5c) is the same as equation (7.3-4).

If the windings are of the same polarity then fluxes $\phi_1$ and $\phi_2$ will add. If they are of opposite polarity then the fluxes will subtract. The consequence is that the mutual inductance, which is the linked
relationship of one winding to the other, will subtract if they are of opposite polarity. The context is indicated by figure 7.3-2.

**Figure 7.3-2.** Relative polarities of the windings with flux direction indicated by $B$-field.

### EXAMPLE 7.3-1: Determine the self and mutual inductances for the following magnetic circuit loop. Assume $N_1 = 50$ and $N_2 = 100$ and that the relative permeability $\mu_R = 5000$,

**SOLUTION:**

$$
\mathcal{R} = \frac{1}{\mu_R \mu_0} \frac{\ell}{A} \approx \frac{1}{4\pi} \times \frac{1}{5000} \times \frac{6\pi}{(\pi/4)^2} \text{nH}^{-1}
$$

$$
= 0.096 \text{ MA/Wb} \approx 0.1 \text{ MA/Wb}
$$

Therefore

$$
L_1 = \frac{N_1^2}{\mathcal{R}} \approx \frac{50^2}{0.1} = 25 \text{ mH}
$$

$$
L_2 = \frac{N_2^2}{\mathcal{R}} \approx \frac{100^2}{0.1} = 100 \text{ mH}
$$

$$
M = \frac{N_1 N_2}{\mathcal{R}} \approx \frac{50 \times 100}{0.1} = 50 \text{ mH}
$$

And notice that

$$
M = \sqrt{L_1 L_2} = \sqrt{25 \times 100} = 50 \text{ mH}
$$

Of course the rest of the story is the electrical relationships defined by

$$
v_1 = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}
$$

$$
v_2 = \pm M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}
$$

(7.3-6a) (7.3-6b)
If sinusoidal current relationships are assumed for \(i_1\) and \(i_2\), i.e.

\[
i = I e^{j\omega t}
\]

Then the derivatives will give the relationship

\[
v_1 = j\omega L_1 \times i_1 \pm j\omega M \times i_2 = sL_1i_1 \pm sMi_2 \quad (7.3-7a)
\]

\[
v_2 = \pm j\omega M \times i_1 + j\omega L_2 \times i_2 = \pm sMi_1 + sL_2i_2 \quad (7.3-7b)
\]

Where \(s = j\omega\). If \(M\) is positive then multiplying equation (7.3-7a) by \(L_2\) and equation (7.3-7b) by \(-M\) and adding the two eliminates \(i_2\). If \(M\) is negative then (7.3-7b) is multiplied by \((+M)\) and the same addition eliminates \(i_2\). The result will be

\[
v_1L_2 - Mv_2 = j\omega(L_1L_2 - M^2)x_{i_1} = 0 \quad \text{since} \quad M = \sqrt{L_1/L_2} \quad (7.3-8)
\]

And therefore

\[
v_1L_2 = Mv_2
\]

or

\[
\frac{v_2}{v_1} = \frac{L_2}{M} = \frac{N_2^2/\Re}{(N_1N_2)/\Re} = \frac{N_2}{N_1} \quad (7.3-9a)
\]

Similarly \(i_1\) could have been eliminated with result

\[
\frac{v_2}{v_1} = \frac{M}{L_1} = \frac{N_2^2/\Re}{N_1^2/\Re} = \frac{N_2}{N_1} \quad (7.3-9b)
\]

Equation (7.3-9) is called the \underline{ideal transformer equation}. It results in a simple transfer function with ‘turns ratio’ \(n = N_2/N_1\). So the transformer finds itself a niche as a relatively common circuit component.

If the transformer is also lossless then \(p_2 = p_1\), which is the same as \(i_2v_2 = i_1v_1\). And therefore

\[
\frac{i_2}{i_1} = \frac{v_1}{v_2} = \frac{N_1}{N_2} \quad (7.3-10)
\]

Equations (7.3-9) and (7.3-10) are the \underline{ideal lossless transformer equations} and are the default first-order mathematical description of transformers.

The circuit symbol for the ideal transformer is shown by figure 7.3-3. As is true for all circuit symbols it is representative of the physical topology of the transformer, such as that shown by figure 7.3-1.
Figure 7.3-3. Transformer circuit symbols. The different forms represent different core options, with the assumption that (b) is an iron-core transformer even though it may be some other highly permeable material (e.g. ferrite, silicon steel).

Figure 7.3-3 also shows the typical representation for the turns ratio \( N_2/N_1 \). The ratio can be represented as a single number \( n (= N_2/N_1) \) or as a left-right number pair \( N_1:N_2 \). Both of these forms are common, regardless of the other aspects of the transformer. The turns ratio \( n \) need not necessarily be an integer.

If \( n \) is greater than 1 then the transformer is called a step-up transformer. If \( n \) is less than 1 it is called a step-down transformer.

In some applications it is necessary to be aware of the transformer polarities defined by the relative polarity of the coil windings. The convention is to make the use of ‘dots’, as indicated by figure 7.3-4(2). The dot is usually only of importance to an output relative to another output winding (we can have more than two output windings). But they are an occasional aspect of any circuit design using transformers.

The rest of the story for the ideal transformer is that impedances external to each side of the transformer also play a role in the currents and voltages that are linked by the transformer. Consequently these impedances will have an equivalent image that ‘reflected’ across the transformer. Due to the current and voltage links the voltage-current ratio on side will relate to the ratio on the other side according to equations (7.3-9) and (7.3-10). This results in a combined relationship of the form

\[
\frac{v_2}{i_2} = \frac{n_{21}v_1}{i_1} = n_{21}^2 \times \frac{v_1}{i_1}
\]

(7.3-11)
where \( n_{21} = N_2/N_1 \) is the (forward) turns ratio. Equation (7.3-11) translates into a relationship for the image of an impedance reflected across the transformer of the form

\[
Z_{21} = n_{21}^2 \times Z_{11}
\]  
(7.3-12)

The subscripts assist in keeping the images in good order. \( Z_{21} \) is the impedance on side 1 (= \( Z_{11} \)) reflected from side 1 to side 2. If the subscripts are flipped then \( Z_{12} \) is the impedance on side 2 reflected to side 1. The order of the impedance subscripts has the same meaning as ‘to-from’.

\[
Z_{12} = n_{12}^2 \times Z_{22}
\]
(7.3-13)

where the subscripts on the turns ratio mean that we have the inverse, i.e.

\[
\frac{N_1}{N_2} = n_{12} = \frac{1}{n_{21}} = \frac{1}{N_2/N_1}
\]
(7.3-14)

The rules of engagement may be stated more concisely as follows:

\[
v_2 = n_{21} \times v_1
\]
(7.3-15a)

\[
i_2 = i_1 \times n_{12}
\]
(7.3-15b)

\[
Z_{12} = n_{12}^2 \times Z_{22}
\]
(7.3-15c)

Take note of the order of the subscripts and that the current factor \( n_{12} \) is represented as a post-multiplier where the others are identified as pre-multipliers. Take note of the context of the subscripts for the transfer factors \( n_{21} \) and \( n_{12} \) relative to the turns ratio represented by equation (7.3-14).

The mathematics obscures the fact that a left-right flow of current and the implication that \( i_2 \) is due to \( i_1 \) also implies that the negative coefficient for \( M \) is the correct option. It also implies that the (ideal) transformer stores no flux and therefore stores no energy, and consequently will not have any inductance at either the primary input or the secondary input other than that reflected across the transformer.

Consider the following example

**EXAMPLE 7.3-2:** An ideal transformer with \( n_{21} = 2.0 \) has impedances \( R_{11} \) and \( R_{22} \) as shown. (labeled as \( R_1 \) and \( R_2 \), respectively). Determine currents and the voltages across the transformer by (a) reflecting \( R_{22} \) to the primary \( (v_1) \) side and (b) by reflecting \( R_{11} \) to the secondary \( (v_2) \) side.

*Figure E7.3-2(a): Example circuit.*
SOLUTION: (a) Since \( n_{21} = 2.0 \) then \( n_{12} = 0.5 \). And so (by equation (7.3-13))

\[ R_{12} = 0.5^2 \times 80 = 20k\Omega \]

and so the circuit equivalent now looks like that of figure E7.3-2(b). The vertical dashed line is the image plane due to the transformer.

The circuit equivalent is of the form of a voltage divider for which (by inspection) \( v_i = 10V \) = voltage on the primary side of the transformer.

The current on the primary side \( i_i = \frac{15V}{(10k + 20k)} = 0.5mA \)

Figure E7.3-2(b): secondary side impedance reflected to primary side.

The current on the secondary side is then \( i_2 = i_i \times n_{12} = 0.5mA \times 0.5 = 0.25mA \)

and so voltage \( v_2 \) on the secondary side of the transformer is:

\[ v_2 = 0.25mA \times 80k\Omega = 20V \]

which also agrees with \( v_2 = n_{21} \times v_i = 2.0 \times 10V \)

(b) If instead \( R_{11} \) is reflected to side 2 then

\[ R_{21} = n_{21}^2 \times R_{11} = 2.0^2 \times 10 = 40k\Omega \]

And the circuit equivalent is of the form of a voltage divider as shown by figure E7.3-2(c). The vertical line represents the image plane seen from the secondary side (side 2).

by inspection of the equivalent voltage divider

\[ v_2 = 20V \]

= voltage on the secondary side of the transformer.

Figure E7.3-2(c): Primary (side 1) impedance reflected to side 2

The current on the secondary side is then

\[ i_2 = \frac{20V}{80k} = 0.25mA \]

and then \( i_i = i_2 \times n_{21} = 0.25mA \times 2.0 = 0.5mA \)

and \( v_i = n_{12} \times v_2 = 0.5 \times 20 = 10V \)

These results are the same as part (a) and the two options verify each other.
The example illustrates how the use of indexed subscripts keeps the order and orientation of impedance reflections across the transformer consistent and simple. The concept of reflections and ‘image planes’ serves to effectively reduce an ideal transformer topology to an equivalent single loop.

One of the more important transformer topologies is the series pair, step-up, step-down transformer topology. It has the effect of significantly increasing the efficiency of power transferred to the load as well as reducing relative power loss due to line impedance. The transformers have the effect of accomplishing impedance matching for efficiency rather than balance. The efficiency improvement is of significant interest and signature value to the AC power grid. It is the basis for high-voltage transmission line systems for long-distance and local grid transmission of 60Hz (or 50Hz) AC power. The concept and construct is shown by figure 7.3-5.

Figure 7.3-5. Step-up, step-down transformer pair

In order to analyze a circuit with two transformers it is in order to make two reflections. There are three options since (1) an image can be formed of the load reflected back to the source. Or (2) the load and the source can be reflected to the middle section. Or (3) an image can be formed of the source and it can be reflected to the load end.

As indicated by the rules of engagement (equations 7.3-15) attention to the indices and the pre-and post-multiplication context for the electrical quantities is of considerable benefit. Take note that voltage ratios only apply to the ports, whereas impedances and currents are reflected from one loop to another. Consequently if the current of any one loop is ascertained then the others will follow immediately, i.e.

\[
\begin{align*}
    i_1 &= i_2 \times n_{21} = \left( i_3 \times n_{32} \right) \times n_{31} = i_3 \times \left( n_{32} \times n_{31} \right) \\
    i_3 &= i_2 \times n_{23} = \left( i_1 \times n_{12} \right) \times n_{23} = i_1 \times \left( n_{12} \times n_{23} \right) \\
    i_2 &= i_3 \times n_{32} \quad \text{or} \quad i_2 = i_1 \times n_{12}
\end{align*}
\]  

(7.3-16a) \hspace{2cm} (7.3-16b) \hspace{2cm} (7.3-16c)

These equations are redundant. The basic rule of reflected currents is given by equation (7.3-15b).

Consider the following example, which will also examine the power efficiency of the step-up step-down construct, for which

\[
\eta = \frac{P_L}{P_S} 
\]  

(7.3-17)
**EXAMPLE 7.3-3:** (Use of step-up/step-down transformers to reduce line loss). Determine (a) all of the currents and voltages for each case (b) power from the source and to the load in each case, and (c) efficiency $\eta = p_L/p_S$ in each case.

**Figure E7.3-3(a):** Source with $20\Omega$ line resistance $R_S$ and matched load ($20\Omega$).

**Figure E7.3-3(b):** Same with balanced up-down transformers and line resistance $R_S$ in the middle stage.

**SOLUTION:** Topology #1 (direct, matched)

- $i = 100/40 = 2.5A$
- $v_L = \text{50 VAC}$ by inspection
- $p_S = 100V \times 2.5 = 250W$
- $p_L = 50V \times 2.5 = 125W$
- $\eta_0 = p_L/p_S = 50\%$

Topology #2: Reflecting everything to the middle stage

- $v_2 = n_{21} \times v_1 = 2.0 \times 100 = 200V$
- $R_{23} = n_{23}^2 \times R_{33} = n_{23}^2 \times R_L = 2^2 \times 20 = 80\Omega$

Then the equivalent middle stage is as represented by figure E7.6-3(c)

- $i_2 = v_2/(R_{22} + R_{23}) = 200/(20 + 80) = 2.0A$
- $i_3 = n_{23} \times i_2 = 2 \times 2.0 = 4.0A$
- $i_1 = n_{21} \times i_2 = 2 \times 2.0 = 4.0A$

**Figure E7.3-3(c):** Reduced up-down topology. The vertical lines represent the transformer interfaces.

- $v_3 = i_3 \times R_{33} = i_3 \times R_L = 4.0 \times 20 = 80V$
- $p_S = i_1 \times v_1 = 4.0 \times 100 = 400W$
- $p_L = i_3 \times v_3 = 4.0 \times 80 = 320W$

and the efficiency is $\eta_0 = p_L/p_S = 320/400 = 80\%$
If (extending the reach) the up/down transformer pair had been $n_{21} = 5$ and $n_{32} = 1/5$

then $P_L = 462W$ and $\eta = 96\%$.

Not only does topology #2 deliver more power to the load, the efficiency can be pushed close to 100%.
The transformer interface has consequently found a good bit of utility in the energy industries and other
energy transfer applications.

Take note of the context and ordering of the subscripts in the transformer transfer ratio, for which the
rules of engagement of equations (7.3-15) are applicable and for which the subscripting takes care of the
reflections from one loop to another.

**EXAMPLE 7.3-4:** (Use of step-up/step-down transformers to reduce line loss.) Determine (a) all of
the currents and voltages (b) power from the source and to the load (c) efficiency $\eta = P_L/P_S$.

![Up-down transformer string with source, line and load resistances](image)

**Figure E7.3-4(a):** Up-down transformer string with source, line and load resistances

**SOLUTION:** Reflecting the resistances to the middle stage

$$R_{23} = n_{23}^2 \times R_{33} = n_{23}^2 \times R_L = 2^2 \times 16 = 64\Omega$$

$$R_{21} = n_{21}^2 \times R_{11} = 2^2 \times 2 = 8\Omega$$

And $v_2 = n_{21} \times v_I = 2.0 \times 20 = 40V$

So the up-down transformer pair reduces to the equivalent reflection circuit

![Reduced topology. The vertical lines represent the equivalent transformer interfaces.](image)

**Figure E7.3-4(b):** Reduced topology. The vertical lines represent the equivalent transformer interfaces.

$$i_2 = \frac{v_2}{(R_{21} + R_{22} + R_{23})} = \frac{40}{(8 + 8 + 64)} = 0.5A$$
Take note that if we have one current we then have them all since
\[ i_3 = i_2 \times n_{23} = 0.5 \times 2 = 1.0 \text{A} \]
\[ i_1 = i_2 \times n_{21} = 0.5 \times 2 = 1.0 \text{A} \]

And if we have all of the currents we also have all of the voltages, i.e.
\[ v_3 = i_3 \times R_{33} = i_3 \times R_L = 1.0 \times 16 = 16 \text{V} \]
\[ v_2' = n_{22} \times v_3 = 2 \times 16 = 32 \text{V} \]
\[ p_S = i_1 \times v_1 = 1.0 \times 20 = 20 \text{W} \]
\[ p_L = i_3 \times v_3 = 1.0 \times 16 = 16 \text{W} \]

and the efficiency is \( \eta = p_S/p_S = 16/20 = 80\% \)

### 7.4 THE NON-IDEAL TRANSFORMER and TRANSFORMER MODELS

The tenets of the ideal transformer are developed from equations (7.3-5) and (7.3-7). They assume that the linked flux represented by mutual inductance \( M \) is 100\%. For the non-ideal transformer that assumption is not reasonable. Equation (7.3-5c) must then be re-qualified as

\[ 2 \cdot \frac{1}{L_1L_2} \]  
(7.4-1)

for which \( k \leq 1 \). Constant \( k \) is defined as the transformer coupling factor.

The turns ratio transfer constant \( n = N_2/N_1 \) was a consequence of either (7.3-9a) or (7.3-9b), both of which were outcomes of \( \left( L_1L_2 - M^2 \right) = 0 \). That condition required that mutual inductance \( M \) be of the form given by flux linkage equation (7.3-4) or equivalently by equation (7.3-5c). But if \( k < 1 \) the condition \( \left( L_1L_2 - M^2 \right) = 0 \) fails and the transformer ‘turns ratio’ transfer constant is no longer valid.

But have no cause to fear. A two-port inductance topology with mathematical equivalence to the flux-linked model is in the offing and is sufficient to accommodate equation (7.4-1) and the rest of the inductance relationships, to include the transformer coupling factor \( k < 1 \). An ideal transformer model may then be appended and its transformer constant applied in the same way as with the ideal model.
The equivalence topology for the transformer inductances is a ‘Tee’ model as shown by figure 7.4-1.

![Figure 7.4-1. Tee network of inductances.](image)

A KVL analysis of figure 7.4-1 gives

\[ v_1 = sL_1i_1 + sL_C(i_1 + i_2) \]  
(7.4-2a)

\[ v_2 = sL_C(i_2 + i_1) + sL_Bi_2 \]  
(7.4-2b)

The substitutions

\[ L_C = M \]  
(7.4-3a)

\[ L_A = L_1 - M \]  
(7.4-3b)

\[ L_B = L_2 - M \]  
(7.4-3c)

will make the ‘Tee’ equations (7.4-2a) and (7.4-2b) mathematically identical to the coupling equations (7.3-7a) and (7.3-7b) except the usage of the common reluctance equations (7.3-5) cannot be assumed. Since they defined the transformer constant \( n_{21} = N_2/N_1 \) it now needs to be an appended factor.

And that can be accomplished by appending an ideal transformer model to the Tee model. The relationship between the inductances \( L_1 \) and \( L_2 \) can then be accommodated by use of the impedance relationships for the ideal transformer model given by equation (7.3-12) and (7.3-13)

\[ Z_{21} = n_{21}^2 \times Z_{11} = a^2 Z_{11} \]

\[ Z_{12} = n_{12}^2 \times Z_{22} = Z_{22} / a^2 \]

where \( a = \) ideal transfer ratio (a.k.a. \( n = n_{21} \)). The assumptions employed with the ideal model can be passed along to the T-model to accommodate the transformer coupling factor \( k \). There are several options that accomplish this requirement. Two of the T-model options are shown by figures 7.4-2a and 7.4-2b.
Equations (7.4-4) are the same as the ‘Tee’ equations (7.4-2) except that (7.4-2) uses the assumption that the construct includes an ideal transformer as shown by figure 7.4-2a. Of key importance is the fact that for equation (7.4-4) the likelihood that \( k < 1 \) is of no consequence.

If \( a = \sqrt{L_2 / L_1} \) is applied to option 1 and it is accepted that \( M = k \sqrt{L_1 L_2} \) then a T-model of the form of figure 7.4-4 will result since

\[
M / a = \left( k \sqrt{L_1 L_2} \right) / \sqrt{L_2 / L_1} = kL_1 \quad (7.4-5a)
\]

\[
L_1 - M / a = (1 - k)L_1 \quad (7.4-5b)
\]

\[
L_2 / a^2 - M / a = L_2 / (L_2 / L_1) - kL_1 = (1 - k)L_1 \quad (7.4-5c)
\]

This model is reasonably physical since \( L_1 \) and \( L_2 \) are defined by equations (7.3-5a) and (7.3-5b) and they confirm turns ratio constant \( a = N_2 / N_1 \).
A slightly less physical model with a simpler outcome can be accomplished if we let $a = \frac{L_2}{M}$ (like equation 7.3-9a) and assume that $M = k\sqrt{L_1 L_2}$, for which, referring to figure 7.4-2:

\[
\begin{align*}
L_2 / a^2 - M / a &= 0 & (7.4-6a) \\
M / a &= M^2 / L_2 = \left( k^2 L_1 L_2 \right) / L_2 = k^2 L_1 & (7.4-6b) \\
L_1 - M / a &= \left(1 - k^2 \right) L_1 & (7.4-6c)
\end{align*}
\]

This option then yields the topology of figure 7.4-5.

\[
\begin{align*}
L_m &= \text{Magnetizing inductance} \\
L_p &= \text{Leakage inductance (of primary coil)}
\end{align*}
\]

Figure 7.4-5 is the generally accepted form, characterized by the magnetizing inductance on the order of $L_m \approx L_1$ and the somewhat smaller leakage inductance as indicated.

**EXAMPLE 7.4-1.** Using the results of example 7.3-1 and a transformer coupling factor $k = 0.90$ determine the magnetizing inductance $L_m$ and the primary leakage inductance $L_p$.

**SOLUTION:** In example 7.3-1 the self-inductance on the primary side was $L_1 = 25\text{mH}$

\[
\begin{align*}
L_m &= k^2 L_1 = 0.9^2 \times 25\text{mH} \approx 20\text{mH} \\
L_p &= \left(1 - k^2 \right) L_1 = \left(1 - 0.9^2 \right) \times 25\text{mH} \approx 5\text{mH}
\end{align*}
\]
A more comprehensive model for the non-ideal transformer will include a leakage inductance on the secondary side and resistance in series and in shunt. This model is an extension of Figure 7.4-5 and is shown by figure 7.4-6 without qualification.

Figure 7.4-6. Comprehensive non-ideal transformer model including parasitic wire resistances $R_p$ and $R_s$ and the core loss resistance $R_m$.

The more comprehensive model shown by figure 7.4-6 hides the fact that iron core transformers provide a stronger permeability link but suffer from losses due to eddy currents in the iron and nonlinearities due to saturation and hysteresis. These effects contribute to power loss in the core and are represented by the core loss shunt resistance $R_m$ as shown. The simulation model for resistance $R_m$ may include several extra parameters to accommodate the non-linear behavior and losses in the core under large levels of current and magnetic field. Transformer cores are usually laminated or formed as Ferrite to inhibit or prevent eddy currents.

Eddy currents are the principal heating mechanism for induction furnaces.

### 7.5 ENERGY STORED IN THE TRANSFORMER

Like any device that employs inductances it is expected that the transformer construct will include storage of energy. The fact that the inductances form a flux linkage changes this dynamic since energy that flows into one port will be offset by the flow of energy out the other port.

The power associated with the port currents and voltages is

$$ p(t) = i_1 v_1 + i_2 v_2 \quad (7.5-1) $$

And if equations (7.3-6a) and (7.3-6b) are applied to (7.5-1) then

$$ p(t) = i_1 \left( L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \right) + i_2 \left( L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt} \right) $$
\[ L_1 \left( i_1 \frac{di_1}{dt} \right) \pm M \left( i_1 \frac{di_2}{dt} + i_2 \frac{di_1}{dt} \right) + L_2 \left( i_2 \frac{di_2}{dt} \right) \]  
(7.5-2)

Stored energy \( w = \int p(t) dt \) is then

\[ w = L_1 \int_0^{i_1} i_1 di_1 + M \int_0^{i_2} d(i_1 i_2) + L_2 \int_0^{i_2} i_2 di_2 \]
\[ = \frac{1}{2} L_1 I_1^2 \pm MI_1 I_2 + \frac{1}{2} L_2 I_2^2 \]  
(7.5-3)

Asserting that subtractive choice for \( M \) is consistent with flux coupled out, then the energy density stored in the transformer is then

\[ w = \frac{1}{2} \left[ L_1 I_1^2 + L_2 I_2^2 \right] - MI_1 I_2 = \frac{1}{2} \left[ \left( I_1 \sqrt{L_1} \right) - \left( I_2 \sqrt{L_2} \right) \right]^2 + 2 \left( I_1 \sqrt{L_1} \right) \times \left( I_2 \sqrt{L_2} \right) - MI_1 I_2 \]

\[ \therefore w = \frac{1}{2} \left[ I_1 \sqrt{L_1} - I_2 \sqrt{L_2} \right]^2 + \left( \sqrt{L_1 L_2} - M \right) I_1 I_2 \]  
(7.5-4)

The first term of equation (7.5-4) = 0 since \( \sqrt{L_2/L_1} = N_2/N_1 \) by equations (7.3-5a) and (7.3-5b) and \( I_2/I_1 = N_1/N_2 \) by equation (7.3-10). So equation (7.5-4) implies that no energy is stored in the transformer unless \( M < \sqrt{L_1 L_2} \) which is the same as the non-ideal assumption \( k < 1.0 \).

This analysis shows that only the non-ideal transformer retains any stored energy. It also confirms that the ideal transformer does NOT store energy, consistent with the earlier assertion thereto. The ideal transformer therefore has no port inductance other than those of the leakage and magnetizing inductances \( L_p \) and \( L_m \).
PORTFOLIO and SUMMARY

Magnetic circuits

\[ \vec{B} = \mu \vec{H} \quad B \text{ in Weber/m}^2 \quad H \text{ in Newtons/Amp-m} \]

Reluctance \[ R = \frac{1}{\mu} \frac{\ell}{A} = \frac{1}{\mu_0\mu} \frac{\ell}{A} \quad \mu_0 = 4\pi \text{nH/cm} \quad 1/\mu_0 = 0.8 \text{MA/Wb} \]

Inductance \[ L = \frac{N^2}{R} \]

Ideal transformer

\[ L_1 = \frac{N_1^2}{R} \quad L_2 = \frac{N_2^2}{R} \quad M = \sqrt{L_1L_2} \]

\[ \frac{v_2}{v_1} = \frac{N_2}{N_1} = n_{21} \quad \frac{i_2}{i_1} = \frac{N_1}{N_2} = n_{12} = \frac{1}{n_{21}} \]

Impedance reflection: \[ Z_{21} = n_{21}^2 \times Z_{11} \quad Z_{12} = n_{12}^2 \times Z_{22} \]

Up-down transformer pair

Reflect to the middle (simplest option)

\[ R_{23} = n_{23}^2 \times R_{31} = n_{21}^2 \times R_L \]
\[ R_{31} = n_{31}^2 \times R_{11} \]
\[ v_{S'} = n_{21} \times v_S \]

\[ i_1 = n_{21} \times i_2 \]
\[ i_3 = n_{23} \times i_2 \]
\[ i_3 = n_{21} \times \left(n_{32} \times i_3\right) = n_{31} \times i_3 \]

and if we have one current we have them all.

Efficiency \[ \eta = \frac{p_L}{p_S} \quad \text{where} \quad p_S = i_1 \times v_1 \quad p_L = i_3 \times v_3 \]
Non-ideal transformer  \( M = k \sqrt{L_1L_2} \)  where 0 < \( k \) < 1

T-model of non-ideal transformer

\( a = \frac{L_2}{M}, \quad M = k \sqrt{L_1L_2} \)

Comprehensive non-ideal transformer model including parasitic wire resistances \( R_p \) and \( R_s \) and the core loss resistance \( R_m \), leakage inductances \( L_s \) and \( L_p \) and magnetizing inductance \( L_m \).