ECE 3614
FUNDAMENTALS OF ENERGY SYSTEMS

Topics covered in ECE 3614 ...

- Single-phase power (review)
- Balanced three-phase power (review)  Appendix A
- Magnetic circuits  Chapter 1
- Single-phase transformers  Chapter 2
- Three-phase transformers  Chapter 2
- Per-unit system  Chapter 2
- DC machines  Chapters 7, 8
- Induction (asynchronous) machines  Chapters 6
- Synchronous machines  Chapters 4, 5

Prerequisites for ECE 3614 ...

- Basic circuit analysis techniques
- Basic electromagnetic theory
- AC analysis (single-phase and three-phase)
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**SINGLE PHASE CIRCUITS**

A *single phase circuit* consists of one sinusoidal source driving a given network at a particular frequency. This circuit can be analyzed in the time domain to determine *instantaneous* currents and voltages or in the frequency domain to obtain *phasor* currents and voltages (complex numbers will be denoted by bold font). The circuit analysis is typically performed in the frequency domain since the frequency domain analysis is mathematically easier than that in the time domain (algebraic equations as opposed to integro-differential equations).

**Example (Single-Phase Circuit)**

**Time Domain**

\[
\begin{align*}
    v(t) &= V_{\text{peak}} \cos(\omega t + \theta_v) \\
    &= \sqrt{2} V_{\text{rms}} \cos(\omega t + \theta_v)
\end{align*}
\]

Kirchoff's Voltage Law (Time domain)

\[
v(t) = i(t)R + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt
\]

**Frequency Domain**

\[
V(\omega) = V_{\text{rms}} \angle \theta_v
\]

\[
\omega = 2 \pi f
\]

Kirchoff's Voltage Law (Frequency domain)

\[
V(\omega) = I(\omega) \left[ R + j\omega L + \frac{1}{j\omega C} \right] = I(\omega)Z(\omega)
\]
\[ I(\omega) = \frac{V(\omega)}{Z(\omega)} = \frac{V(\omega)}{R + j\omega L + \frac{1}{j\omega C}} = \frac{V(\omega)}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \]

If \( V(\omega) = 120 \angle 0^\circ \) V-rms at 60 Hz \([v(t) = 120\sqrt{2} \cos(120\pi t)]\), \( R = 100 \Omega \), \( L = 1 \text{ mH} \) and \( C = 10 \mu\text{F} \), the current in the single phase circuit is

\[ I(\omega) = \frac{120\angle 0^\circ}{100 + j\left((120\pi)(10^{-3}) - \frac{1}{(120\pi)(10^{-5})}\right)} \]

\[ = \frac{120\angle 0^\circ}{283.1 \angle -69.29^\circ} \]

\[ = 0.4239 \angle 69.29^\circ \]

\[ = 0.4239 e^{j69.29^\circ} \text{ A-rms} \]

\[ i(t) = \sqrt{2} \text{ Re}\{I(\omega)e^{j\omega t}\} = \sqrt{2} \text{ Re}\{0.4239 e^{j69.29^\circ} e^{j\omega t}\} \]

\[ = 0.5995 \cos(120\pi t + 69.29^\circ) \text{ A} \]

The instantaneous power for the single-phase circuit is given by

\[ p(t) = v(t)i(t) = [169.7 \cos(120\pi t)][0.5995 \cos(120\pi t + 69.29^\circ)] \]

\[ = 101.7 \cos(120\pi t)\cos(120\pi t + 69.29^\circ) \text{ W} \]

Using the following trigonometric identity for the product of two cosines,

\[ \cos\Phi_1\cos\Phi_2 = \frac{1}{2} \left[ \cos(\Phi_1 - \Phi_2) + \cos(\Phi_1 + \Phi_2) \right] \]

the instantaneous power becomes

\[ p(t) = \frac{101.7}{2} \left[ \cos(69.29^\circ) + \cos(240\pi t + 69.29^\circ) \right] \]

\[ = 17.98 + 50.85 \cos(240\pi t + 69.29^\circ) \text{ W} \]
Note that the instantaneous power \( p(t) \) is represented as the sum of a DC (time-invariant) component and an oscillatory component at twice the frequency of the instantaneous current \( i(t) \) and voltage \( v(t) \). Thus, the instantaneous power delivered to the circuit is oscillatory.

Also note that the instantaneous power \( p(t) \) as defined for the single phase circuit is positive at certain times in the cycle and negative at others. The positive values of \( p(t) \) represent real power being delivered to (dissipated by) the network (load) while negative values of \( p(t) \) represent power being dissipated by the source.
**SINGLE-PHASE INSTANTANEOUS POWER**  
*(General Equation)*

The voltage and current in a single phase circuit can be written in general form as

\[
v(t) = \sqrt{2} V_{\text{rms}} \cos(\omega t + \theta_v)
\]

\[
i(t) = \sqrt{2} I_{\text{rms}} \cos(\omega t + \theta_i)
\]

where \( \theta_v \) and \( \theta_i \) are the phase angles of the voltage and current, respectively. Note that this general representation allows for voltages and currents which vary as \( \sin(\omega t) \) according to

\[
\cos \left[ \omega t - \frac{\pi}{2} \right] = \sin(\omega t)
\]

The instantaneous power is given by

\[
p(t) = v(t)i(t) = 2 V_{\text{rms}} I_{\text{rms}} \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)
\]

Using the trigonometric identity,

\[
\cos\phi_1\cos\phi_2 = \frac{1}{2} \left[ \cos(\phi_1 - \phi_2) + \cos(\phi_1 + \phi_2) \right]
\]

the instantaneous power becomes

\[
p(t) = V_{\text{rms}} I_{\text{rms}} \left[ \cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i) \right]
\]

In terms of peak values, the instantaneous power is

\[
p(t) = \frac{V_{\text{peak}} I_{\text{peak}}}{2} \left[ \cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i) \right]
\]
TIME AVERAGE POWER AND POWER FACTOR

The time average power ($P$) in watts delivered to a circuit is defined as the integral of the instantaneous power over one period, divided by the period, or

$$P = \frac{1}{T} \int_{t_o}^{t_o+T} p(t)dt$$

Inserting the expression for the single phase instantaneous power gives

$$P = \frac{1}{T} \int_{t_o}^{t_o+T} V_{rms} I_{rms} [\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)]dt$$

$$= \frac{V_{rms} I_{rms}}{T} \left\{ \cos(\theta_v - \theta_i) \int_{t_o}^{t_o+T} dt + \int_{t_o}^{t_o+T} \cos(2\omega t + \theta_v + \theta_i) dt \right\}$$

The first integral in the previous equation is equal to the period $T$ while the second integral is equal to zero (the integral of a cosine function over two periods). The time average power delivered to the circuit becomes

$$P = V_{rms} I_{rms} \cos(\theta_v - \theta_i)$$

This equation shows that the time-average power is maximum when the cosine function is unity (when the voltage and current are in phase). The product of $V_{rms}I_{rms}$ is the maximum possible available power and is designated as the apparent power. The cosine function in the average power expression is designated as the power factor ($PF$). According to the average power expression, the power factor is the ratio of average power (real power) to the apparent power.

$$PF = \cos(\theta_v - \theta_i) = \frac{P}{V_{rms}I_{rms}}$$
The power factor defines the time-domain relationship between real and apparent power in terms of the instantaneous voltage and current phase angles. In the frequency domain, we define the concept of complex power using the phasor voltage and current. The complex power can be shown to be directly related to the power factor. Given the phasor voltage and current in an arbitrary circuit defined by

\[
V(\omega) = V_{\text{rms}} \angle \theta_v = V_{\text{rms}} e^{j\theta_v} = V_{\text{rms}} (\cos \theta_v + j \sin \theta_v) = V_{\text{real}} + j V_{\text{imag}}
\]

\[
I(\omega) = I_{\text{rms}} \angle \theta_i = I_{\text{rms}} e^{j\theta_i} = I_{\text{rms}} (\cos \theta_i + j \sin \theta_i) = I_{\text{real}} + j I_{\text{imag}}
\]

we first define the complex power \( S \) by taking the product of the phasor voltage and the complex conjugate of the phasor current.

\[
S = V(\omega)I^*(\omega)
\]

The complex conjugate of the current is

\[
I^*(\omega) = I_{\text{real}} - j I_{\text{imag}} = I_{\text{rms}} (\cos \theta_i - j \sin \theta_i) = I_{\text{rms}} e^{-j\theta_i} = I_{\text{rms}} \angle -\theta_i
\]

Inserting the complex conjugate of the current into the complex power expression gives

\[
S = V(\omega)I^*(\omega) = (V_{\text{rms}} \angle \theta_v) (I_{\text{rms}} \angle -\theta_i)
\]

\[
= V_{\text{rms}} I_{\text{rms}} \angle (\theta_v - \theta_i)
\]

\[
= V_{\text{rms}} I_{\text{rms}} e^{j(\theta_v - \theta_i)}
\]

The complex power can be separated into its real and imaginary components using Euler’s identity.

\[
e^{j(\theta_v - \theta_i)} = \cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)
\]
The complex power becomes
\[
S = V_{rms} I_{rms} \cos(\theta_v - \theta_i) + j V_{rms} I_{rms} \sin(\theta_v - \theta_i) = P + jQ
\]
where the real part of the complex power \(P\) is the same time-average power expression found using instantaneous quantities. The real part of the complex power is commonly referred to as the \textit{real} or \textit{average} power and can be expressed as the product of the apparent power and the power factor.

\[
P = \text{Re}\{S\} = V_{rms} I_{rms} \cos(\theta_v - \theta_i) = V_{rms} I_{rms} \left( PF \right)
\]

The imaginary part of the complex power \(Q\) is commonly referred to as the \textit{reactive} or \textit{quadrature} power.

\[
Q = \text{Im}\{S\} = V_{rms} I_{rms} \sin(\theta_v - \theta_i)
\]

Note that the magnitude of the complex power is the apparent power.

\[
|S| = \left| V_{rms} I_{rms} e^{j(\theta_v - \theta_i)} \right| = V_{rms} I_{rms}
\]

The properties of the complex power and its components can be defined concisely in the complex plane using the \textit{power triangle}. 
The angle of the complex power \( (\theta_v - \theta_i) \) delivered to a particular load is related to the load impedance according to

\[
|S| = V_{rms}I_{rms} \\
P = |S| \cos(\theta_v - \theta_i) \\
Q = |S| \sin(\theta_v - \theta_i) \\
PF = \cos(\theta_v - \theta_i)
\]

Note that the impedance angle is identical to the angle defined in the power factor equation. Thus, this angle is commonly referred to as the power factor angle. Given that the resistance is always positive and the reactance can be positive (inductive) or negative (capacitive), the range of the impedance angle (and the power factor angle) is \((-90^\circ \text{ to } +90^\circ)\). Given that the cosine function is an even function, this leads to an ambiguity in the power factor definition for positive and negative angles. This ambiguity is resolved by designating the power factor as either leading or lagging, according to the phase of the current with respect to the voltage.
Inductive Load (lagging $PF$)

Reactance is positive

Current lags voltage

Capacitive Load (leading $PF$)

Reactance is negative

Current leads voltage

Reactive power is positive

$(\theta_v - \theta_i) > 0$

Reactive power is negative

$(\theta_v - \theta_i) < 0$
Complex Power (units)

The complex power, real power and reactive power all have units equivalent to the product of voltage and current. In order to differentiate between these quantities, we define three separate unit designations for the three components of power.

- Complex power $S$  \[ \text{unit} = \text{VA} \] (Volt-Ampere)
- Real power $P$  \[ \text{unit} = \text{W} \] (Watt)
- Reactive Power $Q$  \[ \text{unit} = \text{VAR} \] (Volt-Ampere -Reactive)

Example (Complex power and power factor)

For the previously defined single phase circuit, determine (a.) the apparent power (b.) the power factor (c.) the real power (d.) the reactive power and (e.) the complex power.

From the previous phasor analysis,

\[ V(\omega) = 120 \angle 0^\circ \text{ V –rms} \]
\[ I(\omega) = 0.4239 \angle 69.29^\circ \text{ A –rms} \]

(a.) \[ |S| = V_{\text{rms}} I_{\text{rms}} \]
\[ = (120)(0.4239) = 50.87 \text{ VA} \]

(b.) \[ PF = \cos(\theta_v - \theta_i) \]
\[ = \cos(-69.29^\circ) \]
\[ = 0.3536 \text{ leading} \]
(c.) \( P = V_{rms} I_{rms} \cos(\theta_v - \theta_i) = (50.87)(0.3536) = 17.98 \) W  

(d.) \( Q = V_{rms} I_{rms} \sin(\theta_v - \theta_i) = (50.87)(-0.9354) = -47.58 \) VAR  

(e.) \( S = P + jQ = (17.98 - j47.58) \) VA  
\[ = VI^* = (120\angle 0^\circ)(0.4239 \angle -69.29^\circ) = 50.87 \angle -69.29^\circ \) VA
**Power Factor Correction**

In a typical power system, the source \( V_s \) is connected to the load \( Z_L \) through some sort of transmission line. The effect of this transmission line (including losses) can be approximated by a simple impedance \( Z_{tl} \) in series with the load as shown below.

\[
\begin{align*}
I & = I_{rms} \angle \theta_i \\
V_L & = V_{rms} \angle \theta_v \\
\end{align*}
\]

If we draw the power triangle for the load, we see that the apparent power \( |S| = I_{rms} V_{rms} \) necessary to deliver a particular value of real power \( P \) to the load increases as the power factor decreases.

Given a constant voltage, the only way to provide the higher apparent power is to supply a larger current which, in turn, increases the transmission line losses. What is necessary is a way of adjusting the \( PF \) to a higher level without introducing additional losses.
The load power factor can be increased by the simple introduction of a parallel reactance which has the opposite sign of the load reactance. This reduces the reactive power and increases the power factor. Since most industrial loads are primarily inductive, capacitive compensation is necessary for power factor correction.

**Example (Power factor correction)**

A small manufacturing plant consumes 120kW at 0.85 lagging power factor (60 Hz). The voltage at the load is 480\( \angle 0^\circ \) V-rms and the line impedance is assumed to be \( Z_{tl} = (0.5+j1) \) \( \Omega \). Determine (a.) the phasor voltage and the power factor at the transmission line input \([V_s \text{ and } PF_s]\) (b.) the parallel capacitance required to raise the power factor to 0.95 lagging.

(a.) \( \cos(\theta_v - \theta_i)_L = 0.85 \) \( \Rightarrow \) \( (\theta_v - \theta_i)_L = \cos^{-1}(0.85) = 31.79^\circ \)

\[
\begin{align*}
|S_L| &= \frac{P_L}{PF_L} = \frac{120}{0.85} = 141.2 \text{ kVA} \\
S_L &= 141.2 \angle 31.79^\circ \text{ kVA}
\end{align*}
\]

\[
S_L = V_L I^* \quad \Rightarrow \quad I = \frac{S_L^*}{V_L^*} = \frac{141.2 \times 10^3 \angle -31.79^\circ}{480 \angle 0^\circ} = 294.2 \angle -31.79^\circ
\]
The voltage drop across the transmission line $V_{dl}$ is given by

$$ V_{dl} = IZ_{dl} = (294.2 \angle -31.79^\circ)(1.118 \angle 63.44^\circ) = 328.9 \angle 31.65^\circ $$

The voltage at the source is the sum of the load voltage and the voltage across the transmission line.

$$ V_s = V_L + V_{dl} $$

$$ V_L = 480 \angle 0^\circ = 480 + j0 $$

$$ V_{dl} = 328.9 \angle 31.65^\circ = 280.0 + j172.6 $$

$$ V_s = 760.0 + j172.6 = 779.4 \angle 12.80^\circ \text{ V–rms} $$

The power factor at the source is defined in terms of the phase difference between the current and the voltage at the source.

$$ (\theta_v - \theta_i)_s = 12.80^\circ - (-31.79^\circ) = 44.59^\circ $$

$$ PF_s = \cos(\theta_v - \theta_i)_s = \cos(44.59^\circ) = 0.712 \text{ lagging} $$
Thus, the required change in the reactive power is

\[ \Delta Q = Q_L - Q'_L = 74.37 \text{ kVAR} - 39.43 \text{ kVAR} = 34.94 \text{ kVAR} \]

The reactive power provided by the capacitor is defined by

\[
\begin{align*}
S_C &= V_C I_C^* = V_L \frac{V_L^*}{Z_C^*} = \frac{|V_L|^2}{Z_C^*} = -j34.94 \text{ kVAR} \\
Z_C &= \frac{|V_L|^2}{S_C^*} = \frac{480^2}{j34.94 \times 10^3} = \frac{1}{j\omega C} \\
C &= \frac{34.94 \times 10^3}{(480^2)(377)} = 402.3 \mu F
\end{align*}
\]