CHAPTER 20. NOISE ANALYSIS and LOW NOISE DESIGN

20.1 THE ORIGINS OF NOISE

Electrical noise is a background “grass” of unwanted signals, usually due to thermal origins. It has a nearly constant amplitude density across the frequency spectrum that tends to mask and obscure the waveforms and information which we wish for our circuits to process.

Noise is an inescapable fact of circuits and signals. It is generated in most part by thermal fluctuations in the motion and flow of charge. It is an important factor in the design and analysis of communication circuits, and therefore most treatments of noise are developed in the context of communications electronics. But electrical noise, and its companion problem, distortion, are an important and necessary consideration of any circuit, in which a signal, whether of linear or logic form, is to be processed.

A figure of merit that defines a circuit in terms of its signal transfer properties is the dynamic range (DR) given by

\[ DR = \frac{\text{largest usable signal}}{\text{smallest usable signal}} \]

The smallest usable signal level is defined by the *noise limit*. The largest usable signal is defined by the *distortion limit*, which is usually a consequence of the compliance (±V_S) limits of the circuit.

In matters of electrical noise, components and devices are defined by thermal kinetics. Thermal statistical fluctuations will produce a random set of signals within an electrical component. Thermal effects manifest themselves as fluctuations in electrical currents. The basic unit of thermal energy (fluctuation) is given by \( \Delta w = kT \) (defined by the fugacity for electrons), where \( k = \) Boltzmann’s constant and \( T = \) absolute temperature. Translate that into the fluctuations in power and

\[ \Delta P = \Delta w \times B = kT \times B \]  \hspace{1cm} (20.1-2a)

where \( B = \) bandwidth of the frequencies passed by the circuit. Since power usually relates to signal through resistance then the (rms) noise thermal power is

\[ \Delta P = \frac{e_n^2}{R} = \frac{i_n^2}{G} \]  \hspace{1cm} (20.1-2b)

where \( R = \) resistance and \( G = \) conductance. These give noise signals in terms of \( e_n = \) noise voltage and \( i_n = \) noise current. The maximum signal that can be passed from the source, \( = \) the resistance itself, (according to maximum power transfer) is then
Equation (20.1-3) is called Johnson-Nyquist's theorem. Note that the measure is in terms of a mean square noise voltage (or current).

Consider the following example:

**EXAMPLE 20.1-1:** What is the amplitude $e_n$ of the thermal noise associated with a resistance $R = 1\, \Omega$ at a noise bandwidth $B = 1\, \text{MHz}$?

**SOLUTION:**

$$e_n^2 = 4kTRB = 4 \times \frac{kT}{q} \times q \times R \times B$$

$$= 4 \times .025 \times (1.602 \times 10^{-19}) \times 10^3 \Omega \times 10^6 \text{s}^{-1}$$

$$= (1.602 \times 10^{-20}) \times 10^9 \text{V}^2 = 16 \times 10^{-12} \text{V}^2$$

The calculation assumes that the circuit is at room temperature (~300K) for which $kT/q = V_T (= \text{thermal voltage}) \approx .025\text{V}$.

The $4kT$ term has value $4kT = 1.6 \times 10^{-20} \text{ (room temperature)}$ and this value may be adopted as a shortcut default for most situations involving noise analysis.

Taking the square root gives noise amplitude

$$e_n = 4\mu\text{V}$$

Equations (20.1-3a) and (20.1-3b) may be represented by the network equivalents of figure 20.1-1

![Figure 20.1-1: Equivalent circuits for thermal noise.](image)

This type of noise is also called Johnson noise, or white (full spectrum) noise since it is spread uniformly across all frequencies of the spectrum. It is also called thermal noise since it is a result of thermal statistics.
Even though each resistance contributes noise as its own little noise source and the network can be analyzed as if it were a collection of noise sources and resistances, this would be a challenging and totally useless exercise. The only place that it finds merit is in circuit simulation, where a part-by-part analysis might be appropriate. Use of Thevenin theorem to reduce the network to a single source and resistance works just fine. The only reason that the separated analysis might be of benefit is for sensitivity analysis of each resistance, and that is a task that is more appropriate for circuit simulation software.

Assessment of a network with both resistances and reactive components implies the need for an expanded form of equation (20.1-3). This expanded form is the *Nyquist formula* and is given by

\[ e_n^2 = 2kT \int_{-\infty}^{+\infty} R(f) df = 4kT \int_{0}^{+\infty} R(f) df \]  

(20.1-4)

where \( R(f) \) is the real part of the impedance \( Z(f) \), as seen across the output nodes.

The Nyquist formula can also be written as

\[ e_n^2 = 4kTR \int_{0}^{+\infty} G(f) df = 4kTRB \]

for which \( G(f) \) is the (frequency dependent) power gain = \( v_{out}/v_{in} \), assuming the resistance term \( R \) is separable. And it gives a clear definition of the noise bandwidth.

Consider the following example:

**EXAMPLE 20.1-2:** Consider the single-time constant \( RC \) network shown by figure 20.1-2. Using the Nyquist formula find (a) noise signal \( e_n^2 \) and (b) noise bandwidth \( B \).

![Figure 20.1-2: Single-time constant RC network.](image)

**SOLUTION:** The impedance looking into the output is \( Z = \frac{1}{G + sC} = \frac{R}{1 + j\omega RC} \)

and its real component is \( \text{Re}[Z(f)] = \frac{R}{1 + \omega^2 R^2 C^2} = R(f) \)
The Nyquist formula then gives

\[ e_n^2 = 4kT \int_0^\infty \frac{R}{1 + \omega^2 RC^2} df = \frac{4kT}{2\pi C} \int_0^\infty \frac{RCd\omega}{1 + \omega^2 RC^2} = \frac{4kT}{2\pi} \times \frac{1}{RC} \times \frac{\pi}{2} = \frac{4kTR}{2\pi RC} \times \frac{\pi}{2} \]

and the noise bandwidth is

\[ B = \frac{\pi}{2} \times f_{3db} \]

Note that the noise bandwidth is greater than the (STC) 3dB bandwidth by the factor \( \pi/2 \). Extending this outcome as an approximation to all circuit networks is not unreasonable and so noise bandwidth will then be expected to relate to the circuit (3dB) bandwidth as:

\[ \text{Noise bandwidth} = B = \frac{\pi}{2} \times f_{3db} = \frac{1}{4} \omega_{3db} \quad (20.1-5) \]

### 20.2 SYSTEM NOISE ANALYSIS

Noise is a form of background signal. In this respect it should be expected that the detector/transducer at the front end of the circuit is the major source of the noise, in terms of some sort of ‘dark current’ signal, with characteristics as might be represented by the source resistance \( R_S \). But it is also true that the components within the amplifier/interface circuit also contribute noise, and it is the assessment of the system noise factors that is necessary to identifying the noise characteristics of the system.

The quantitative comparison between carrier signal and noise signal is defined by the signal-noise ratio

\[ \frac{S}{N} = \frac{\text{(signal power)}}{\text{(noise power)}} \quad (20.2-1) \]

Both signal \( S \) and noise \( N \) are amplified by the gain factor \( G \) of the amplifier. But the noise \( N_O \) at the output is greater than the amplified input noise \( = G \times N_i \) courtesy of the additional noise contributed by the components within the amplifier, i.e.

\[ N_O = GN_i + N_{Oa} = GN_i + GN_a = G(N_i + N_a) \quad (20.2-2) \]
where $N_{Oa}$ = additional noise at the output = $GN_a$. It is as if an additional noise source $N_a$ exists at the input. The sum ($N_i + N_a$) is then an equivalent noise input, and it gives a means to define the principal noise characteristic for the system $\equiv$ the noise factor $F$, identified by the ratio

$$F = \frac{N_i + N_a}{N_i} = 1 + \frac{N_a}{N_i} \quad (20.2-3)$$

Take note that $F$ is always greater than unity.

Equation (20.2-3) can be restated in terms of the power gain $G = S_O / S_i$, where $S_O$ and $S_i$ represent the signal levels at output and input, respectively of the system, i.e.

$$F = \frac{N_O}{GN_i} = \frac{N_O}{(S_O / S_i)N_i} = \frac{S_i}{N_i} \quad (20.2-4)$$

i.e. the noise factor also represents the degradation of the $S/N$ due to the system.

It is important to note that the signal-to-noise and noise factor are ratio relationships of signal power. The maximum signal power transferred and consequently the maximum available noise power per bandwidth occurs when $R_{in} = R_S$, for which

$$N_i = \frac{e^2}{4R_S} = \frac{4kTB}{4R_S} = kTB \quad (20.2-5)$$

and

$$F = 1 + \frac{N_a}{kTB} = 1 + \frac{kT_eB}{kTB} = 1 + \frac{T_e}{T} \quad (20.2-6)$$

Equation (20.2-6) shows that the noise character of a system has a corollary parameter in the form of an equivalent noise temperature $T_e$ for which

$$T_e = (F - 1) \times T \quad (20.2-7)$$

**EXAMPLE 20.2-1:** Noise factor $F = 2.5$ corresponds to

$$T_e = (2.5 - 1) \times 300 = 450K$$

(The nominal circuit temperature $T$ is assumed to be = 300K.)
Since the ultimate interest is the dynamic range of the system, a lower limit, or smallest usable signal, must be identified. This limit is defined as the noise floor \( N_f = S_i(\text{min}) \) needed to achieve a given output \( S/N \). A corollary to this definition is the minimum detectible signal, which is the input signal voltage \( v_i(\text{min}) \) associated with the noise floor. Assuming that \( S_i = v_i^2 / R_s \), the minimum detectible signal is then

\[
v_i(\text{min}) = \sqrt{4R_s S_i(\text{min})}
\]  

(20.2-8)

From the definition (20.2-4) of noise factor in terms of \( S/N \) ratios, and assuming the systems are matched, (for which \( R_{in} = R_s \)) then

\[
N_f = S_i(\text{min}) = N_i \times F \times \frac{S_O}{N_O} = kT \times F \times \frac{S_O}{N_O}
\]  

(20.2-9)

**EXAMPLE 20.2-2:** What is the noise floor and minimum detectible signal for a system with bandwidth 20kHz, noise factor 8dB, input resistance \( R_{in} = 50 \Omega \) if the output signal/noise requirement is 10dB.

**SOLUTION:** Since there is a mix of very large and very small measures in exponent form as well as units of dB it is sometimes advantageous to use logarithms, in which case (20.2-9) becomes

\[
10 \log S_i(\text{min}) = 10 \log(kT) + 10 \log F + 10 \log(S_O / N_O)
\]

\[
= -204 + 33 + 8 + 10 = -143 \text{ dBW}
\]

The measure of dBW is the dB (= order-of-magnitude) measure relative to 1.0W. As a corollary to the logarithmic approach the noise factor \( F \) is often expressed in terms of dB and in some references this usage is called the ‘Noise Figure’ and is abbreviated as ‘NF’.

With the use of logarithmic analysis it is usually convenient to make use of the shortcut

\[
\log(kT) = 10 \log(0.4 \times 10^{-20}) = -204 \text{ dBW}
\]

The value for the noise floor is then either in terms of dBW, or with respect to the 1 milliwatt reference level (= dBm). For this option we would have noise floor value

\[
S_i(\text{min}) = N_f = -143 \text{ dBW} = -113 \text{ dBm}
\]

This may be more dialogue than usual but it is necessary to become acquainted with the units of measure associated with very low RF power levels, as should be expected when speaking the language of noise.

The power level of the noise floor (= \( N_f \)) in watts (via the antilog operation) is then

\[
S_i(\text{min}) = 5 \times 10^{-15} \text{ W} = 5.0 \text{ fW}
\]
for which, using equation (20.2-8) gives minimum detectible signal

\[ v_i(\text{min}) = \sqrt{4R_S s_i(\text{min})} = \sqrt{4 \times 50 \times (5 \times 10^{-15})} = 1 \times 10^{-6}V = 1.0\mu V \]

**Note:** There is nothing wrong with using non-logarithmic arithmetic. In this case it was applied in order to highlight and identify the logarithmic units of measure, dBW and dBm.

### 20.3 CASCADED STAGES

A signal conditioning circuit, particularly those associated with RF applications, generally consists of several stages in cascade to include the antenna, cable(s) as well as those associated with signal gain. Each of these stages contributes noise and has a noise factor and a gain (or loss) factor. The aggregate effect of the system is an equivalent system with input noise \( N_o \) and noise factor as identified by equation (20.2-3).

A system can be interpreted as a cascaded set of stages as represented by figure 20.3-1. The collective noise character can then be resolved in terms of the noise and gain characteristics of each stage by a systematic approach which reflects each contribution back to the input.

![Figure 20.3-1: System as a set of cascaded stages.](image)

For the first stage the equivalent input noise due to its characteristics specified by equation (20.2-3) is

\[ N_{\text{a1}} = N_i(F_1 - 1) \quad (20.3-1) \]

Similarly, the equivalent input noise to the second stage due to its noise character is \( N_{\text{a2}} = N_i(F_2 - 1) \). And when this noise is related to the input of stage one, it can be assumed to be equivalent to an input noise at this point of value
Similarly the equivalent input noise due to \( N_{a3} \) as if it were an equivalent source at the input would be

\[
N'_{a3} = \frac{N_{a3}}{G_4 G_2} = \frac{N_i (F_3 - 1)}{G_1 G_2}
\]

and therefore the collective equivalent input noise due to the three stages would be

\[
N_a = N_{a1} + N'_{a2} + N'_{a3} = N_i (F_1 - 1) + \frac{N_i (F_2 - 1)}{G_1} + \frac{N_i (F_3 - 1)}{G_1 G_2}
\]

(20.3-2)

Since \( F = (N_i + N_a)/N_i \), then the overall system noise figure is

\[
F = F_1 + \frac{(F_2 - 1)}{G_1} + \frac{(F_3 - 1)}{G_1 G_2}
\]

(20.3-3)

In like manner the process can be extended to any number of stages:

\[
F = F_1 + \frac{(F_2 - 1)}{G_1} + \frac{(F_3 - 1)}{G_1 G_2} + \frac{(F_4 - 1)}{G_1 G_2 G_3} + \cdots
\]

(20.3-4)

and since \( T_{e1} = (F_1 - 1) T \), etc, then equation (20.3-4) can be restated in terms of noise temperature, for which

\[
T_e = T_{e1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2} + \frac{T_{e4}}{G_1 G_2 G_3} + \cdots
\]

(20.3-5)

The evaluation of cascaded stages may be a little simpler if noise temperatures are employed.

Consider the following example:

**EXAMPLE 20.3-1:** A receiving antenna that has noise temperature 450K is connected to an amplifier that has noise figure 3dB and gain 10dB by a UHF cable that suffers a loss of 0.2dB per m and has noise figure 2dB. The cable is 5m long.

(a) What is the overall system noise factor and (noise) temperature of the system?
(b) What is the noise factor and temperature of the system if a low noise preamplifier of gain 6dB is and \( F = 1.25 \) is placed right after the antenna (preceding the connecting cable).
ANALYSIS:

(a) Antennas have no gain or loss. They just pick up signal. So the gain of the antenna $G_1 = 1$. If the antenna is not particularly selective it will pick up extraneous background of signals that will be manifested as a noise temperature, which in this case $T_1 = 450\, \text{K}$ and is the same as noise factor $F_1 = 1 + 450/300 = 2.5$.

The cable has thermal properties that manifest themselves as a noise. A noise figure of 2dB corresponds to noise factor $F_2 = 1.585$ [note: $F = \log^1(2/10)$] which corresponds to noise temperature $T_2 = (1.585 - 1) \times 300 = 175\, \text{K}$.

The cable has a loss of $5\, \text{m} \times (-0.2\, \text{dB/m}) = -1\, \text{dB}$, which corresponds to gain factor $G_2 = 0.79$.

It is usually in order to determine noise factor $F$, noise temperature $T_e$, and gain (loss) of each part of a system.

The amplifier has noise factor $F_3 = 2.0$ (same as noise figure 3dB) (same as noise temperature $300\, \text{K}$) and gain $G_3 = 10$.

Listing these in the order indicated:

\begin{align*}
T_1 &= 450, \quad F_1 = 2.5, \quad G_1 = 1.0 \\
T_2 &= 175, \quad F_2 = 1.585, \quad G_2 = 0.79 \\
T_3 &= 300, \quad F_3 = 2.0, \quad G_3 = 10
\end{align*}

The overall noise factor is then

$$F = 2.5 + \frac{(1.585 - 1)}{1.0} + \frac{(2.0 - 1)}{1.0 \times 0.79} = 4.35$$

which is the same as $T_e = (4.35 - 1) \times 300 = 1005\, \text{K}$.

It is usually simpler to find $T_e$ (rather than $F$), i.e.

$$T_e = 450 + \frac{175}{1} + \frac{300}{1.0 \times 0.79} = 1005\, \text{K}$$

(b) For preamplifier of gain $G = 4.0$ (same as 6dB) and noise factor $F = 1.25$ ($T_e = 75\, \text{o}$) inserted at the antenna the system becomes a cascade of four stages, with listings

\begin{align*}
T_1 &= 450, \quad F_1 = 2.5, \quad G_1 = 1.0 \\
T_2 &= 75, \quad F_2 = 1.25 \quad G_2 = 4.0 \\
T_3 &= 175, \quad F_3 = 1.585, \quad G_3 = 0.79 \\
T_4 &= 300, \quad F_4 = 2.0, \quad G_4 = 10
\end{align*}
for which

\[
T_e = 450 + \frac{75}{1} + \frac{175}{1 \times 4} + \frac{300}{1.0 \times 4 \times 0.79} = 662K
\]

= a much lower noise temperature (or noise factor) than before

The example shows that an effective means of improving the S/N of the system (lowering the noise temperature) is to insert a low-noise preamplifier at the front end.

20.4 ANALYSIS AND DESIGN OF LOW-NOISE ELECTRONICS

In the assessment of the noise contributed by a system it is treated as if it is an equivalent input signal, with the system itself as then being little more than a gain element. This equivalence provides the more direct means to identify and quantify the effects of system noise as given by noise factor and/or noise temperature.

System noise needs to be realized both in terms of current fluctuations and voltage fluctuations for \(e_n^2\) and \(i_n^2\) as represented by figure 20.4-1. Both are necessary since system noise will exist even with a short circuit or an open circuit of the source.

Another nomenclature for noise contributions is also represented by figure 20.4-1, namely that of the noise-spectral density, (NSD) i.e. noise signal per bandwidth in either \(V^2/Hz\) or \(A^2/Hz\). This context is not necessary with resistances, since such noise is adequately and succinctly represented by equation (20.1-3), but it is necessary with transistors, since they are dominated by shot noise, which is a noise/Hz. Shot noise has a noise spectral density (NSD) of the form

\[
\langle i_n^2 \rangle = 2qI
\]  

(20.4-1)

and usually is not uniform over the frequency spectrum. It usually is greater at low frequencies and is then usually called pink noise (in comparison to that of thermal noise, which we also call white noise).
Shot noise is distinct from thermal noise since it only occurs when current is flowing. Thermal noise doesn’t care if current is flowing or not. Shot noise contributions vary as $1/f$ and are modeled in circuit simulators by

$$\langle i_{nf}^2 \rangle = K_f \frac{I^a}{f^b} \Delta f$$

(20.4-2)

This type of noise model is supported by the spice simulation utility. Usually $b = 1$. Spice uses AF for $a$ and KF as indicated.

Revisiting resistance noise as an NSD contribution is a matter of dividing equation (20.1-3) by noise bandwidth $B$, for which

$$\langle e_n^2 \rangle = 4kTR$$

(20.4-3a)

$$\langle i_n^2 \rangle = 4kTG$$

(20.4-3b)

Circuit simulators use NSD since noise is an AC analysis exercise for which frequency is the appropriate parameter. The spice nomenclature for $\langle e_n^2 \rangle$ is $NTOT(RS)$. And a spice exercises to this effect accompanies these notes.

Via the NSD nomenclature spice can be used to break out the system noise for a circuit and tailor its design for minimum noise contribution to the system.

Notice that the noise factor $F$ is not dependent on bandwidth, since $B$ is a common factor to both numerator and denominator as identified by equation (20.2-3), for which

$$F = \frac{N_i + N_o}{N_i}$$

Applying the model of figure 20.4-4 for break out of $N_o$ into system-level terms, then

$$N_o = \langle e_n^2 \rangle + \langle i_n^2 \rangle R_S^2$$

(20.4-4)

for which the noise factor becomes

$$F = \frac{4kTR_S + \langle e_n^2 \rangle + \langle i_n^2 \rangle R_S^2}{4kTR_S} = 1 + \frac{\langle e_n^2 \rangle + \langle i_n^2 \rangle R_S^2}{4kTR_S}$$

(20.4-5)

Equation (20.4-5) is the signature element for system analysis and for resolving the effect of the noise contributions. It also shows that $F$ is strongly dependent on $R_S$, with minimum $F$ at

$$\frac{\partial F}{\partial R_S} = \frac{1}{4kT} \left[ -\frac{\langle e_n^2 \rangle}{R_S^2} + \frac{\langle i_n^2 \rangle}{R_S^2} \right] = 0$$
for which, with \( [ ] = 0 \),

\[
R_s(\text{optimum}) = \sqrt{\frac{e_n^2}{i_n^2}}
\]  

(20.4-6)

**EXAMPLE 20.4-1:** Opamps are packaged IC components characterized much like figure 20.4-1. Their noise characteristics are frequency dependent. As such they lend themselves to a log-log graphical representation of \( \{e_n^2\} \) and \( \{i_n^2\} \) or square roots thereto, as represented by figure E20.4-1 for the HA2406 general-purpose opamp:

![Figure E20.4-1: Noise characteristics of the HA2406 general purpose opamp](image)

The noise chart is laid out as follows:

The left-hand axis identifies the NSD for equivalent noise voltage \( \{e_n\} = \sqrt{e_n^2} \) (with units of \( V/\sqrt{Hz} \)).

The right-hand axis identifies the NSD for equivalent noise current \( \{i_n\} = \sqrt{i_n^2} \) (with units of \( A/\sqrt{Hz} \)).

For higher frequencies than the 100kHz boundary, \( \{e_n\} \) and \( \{i_n\} \) will be approximately the same as the right-most values of the chart.

**REQUIREMENT:**

(a) Find optimum source resistance \( R_s(\text{opt}) \) for an application that uses the HA2406 at carrier frequency 20kHz

(b) Noise factor \( F \) and noise temperature \( T_e \) associated with this \( R_s \).

(c) Find \( F \) and \( T_e \) if \( R_s = 50k\Omega \). Instead of \( R_s(\text{opt}) \)

**SOLUTION:**

(a) At the 20kHz mark of figure E20.4-1 \( \{e_n\} \approx 20nV/\sqrt{Hz} \) and \( \{i_n\} \approx 0.85pA/\sqrt{Hz} \).

(Since these have to be visually extracted from a logarithmic scale the values will be inexact.)
and \( R_s (\text{opt}) \approx \frac{20 \text{nV} / \sqrt{\text{Hz}}}{0.085 \text{ pA} / \sqrt{\text{Hz}}} \approx 235 \text{ k}\Omega \)

(b) For this value of \( R_s \), \( 4kTR_s = 1.6 \times 10^{-20} \times 235 \times 10^3 = 37.6 \times 10^{-16} \) and the noise factor is then

\[
F = 1 + \frac{20^2 \times 10^{-18} + (0.085 \times 235)^2 \times 10^{-18}}{37.6 \times 10^{-16}} = 1 + \frac{4 + 4}{37.6} = 1.21
\]

which corresponds to a noise temperature of \( T_e = (1.21 -1) \times 300 = 63 \text{K} \)

(c) For \( R_s = 50 \text{k}\Omega \), \( 4kTR_s = 1.6 \times 10^{-20} \times 50 \times 10^3 = 8.0 \times 10^{-16} \)

using the same values for \( \{e_n\} \) and \( \{i_n\} \) as before, the noise factor is then

\[
F = 1 + \frac{20^2 \times 10^{-18} + (0.085 \times 50)^2 \times 10^{-18}}{8.0 \times 10^{-16}} = 1 + \frac{4 + 0.18}{8.0} = 1.52
\]

which corresponds to a noise temperature of \( T_e = (1.52 -1) \times 300 = 156 \text{K} \)

As is also reflected by equation (20.4-5) a circuit topology can be resolved in terms of equivalent input noise values for \( \{e_n^2\} \) and \( \{i_n^2\} \). Transistors have shot and parasitic thermal noise, and include bias resistances, each of which will have thermal noise.

The shot noise for a BJT is identified in terms of separate contributions due to its base current \( I_B \) and its collector current \( I_C \) as

\[
\{i_{nB}^2\} = 2qI_B \quad \text{(20.4-7a)}
\]
\[
\{i_{nC}^2\} = 2qI_C \quad \text{(20.4-7b)}
\]

In addition to these noise contributions the BJT has a series parasitic resistance at the base node called the base-spreading resistance \( r_B \), for which

\[
\{e_{nB}^2\} = 4kTR_B \quad \text{(20.4-8)}
\]

For the FET, shot noise relates to the square root of the drain current, and consequently (and most usually) is expressed in terms of the transconductance, \( g_m \)

\[
\{i_{nD}^2\} = \frac{2}{3} \times (4kT) \times g_m \quad \text{(20.4-9)}
\]
Circuit simulation assess the noise contributions on a one-by-one, part by part approach and represents each in terms of an NSD equivalent at the input. For the analytical assessment it is more appropriate to identify the \( \{e_{n}^{2}\} \) at the output and reflect it back to the input. The second approach is applied on a per-circuit basis, as represented by the simplified BJT circuit of figure 20.4-3.

![Figure 20.4-3: Common-emitter (CE) topology of the BJT, simplified. The base-spreading resistance \( r_{B} \) is sufficiently small that it will not appreciably affect the circuit transfer characteristics and so it is shown as an ‘x’. \( r_{B} \) does contribute significantly to the noise.](image)

The common-emitter (CE) topology is usually the first case for consideration in the assessment of single transistor topologies. It is also gives a form for which a relatively simple small-signal equivalent can be invoked, as represented by figure 20.4-4a.

![Figure 20.4-4a: Small-signal equivalent of figure 20.2-3.](image)

![Figure 20.4-4b: Small-signal equivalent of figure 20.2-3 with hidden noise sources exposed. For convenience \( R_{d} || R_{L} \) has been replaced by equivalent (resistance) load \( R_{L}' \).](image)
Had more bias frame components been included in figure 20.4-4b it would have been an impressive mess. But otherwise figure 20.4-4b is sufficient to identify the NSD at node $C (\equiv N_{OC})$ as

$$N_{OC} = 4kTR_L + \{i_{nc}^2\} \times (R'_L)^2 = 4kTR_L + 2qI_C \times (R'_L)^2 \quad (20.4-10a)$$

And as represented by the figure the NSD at node B ($\equiv N_{ib}$) is

$$N_{ib} = 4kTr_B + \{i_{nB}^2\} \times (R_S + r_B)^2 \cong 4kTr_B + 2qI_B \times R_S^2 \quad (20.4-10b)$$

where it is expected that base-spreading resistance $r_B << R_S$, so that its effect on the transfer function for the circuit can be neglected. Transistor resistance term $r_\pi$ is a slope and does not generate any noise, but the base-spreading resistance is a small but real resistance.

The total system noise equivalent system NSD at the input is then

$$N_a = N_{ib} + \frac{N_{OC}}{A^2} \cong \left(4kTr_B + N_{OC}/A^2\right) + \{i_{nB}^2\} \times R_S^2 \quad (20.4-11)$$

Comparing this equation to equation (20.4-4) gives

$$\{i_{nC}^2\} = \{i_{nB}^2\} \quad (20.4-12a)$$

$$\{e_x^2\} = \left(4kTr_B + N_{OC}/A^2\right) \cong \left(4kTr_B + \left[4kTR'_L + 2qI_C \times (R'_L)^2\right]/A^2\right) \quad (20.4-12b)$$

The merit of this analysis is that noise and transfer terms for the transistor can be resolved in terms of the current $I = I_C$, i.e.

$$\{i_{nC}^2\} = 2qI_C \quad \{i_{nB}^2\} = 2qI_B = 2q \frac{I_C}{\beta_F} \quad g_m = \frac{I_C}{V_T}$$

and when applied to equation (20.4-11), combined with (20.4-12), and the fact that $A_V = g_mR_L$, then

$$N_a = 4kTr_B + \{i_{nC}^2\} \times R_S^2 + \left[4kTR'_L + \{i_{nc}^2\} \times (R'_L)^2\right]/g_m^2 (R'_L)^2$$

$$= 4kTr_B + 2q \frac{I_C}{\beta_F} R_S^2 + \left[4kTR'_L + 2qI_C \times (R'_L)^2\right]/g_m^2 (R'_L)^2 \quad (20.2-13)$$

This expression simplifies to
\[ N_a = 4kT R_b + 2q \frac{I_C}{\beta_F} R_S^2 + 2q \left( \frac{2kT}{q} \frac{V_T^2}{I_C^2 R_L} + \frac{V_T^2}{I_C} \right) \]
\[ = 4kT R_b + 2q \left( \frac{I_C}{\beta_F} R_S^2 + 2 \frac{V_T^2}{I_C^2 R_L} + \frac{V_T^2}{I_C} \right) \approx 4kT R_b + 2q \left( \frac{I_C}{\beta_F} R_S^2 + \frac{V_T^2}{I_C} \right) \]

for which the system noise \( N_a \) is minimized when

\[ \frac{\partial N_a}{\partial I} = 2q \left[ \frac{R_S^2}{\beta_F} - \frac{V_T^2}{I_C^2} \right] = 0 \]

And for \( [\ ] = 0 \), the optimum current \( I_C \) is then defined by

\[ I_C^2 \approx \frac{\beta_F V_T^2}{R_S^2} \]  \hspace{1cm} (20.2-14)

**EXAMPLE 20.4-2:** Design of a low-noise single-transistor (simplified) input stage.

Determine:  
(a) equivalent input noise terms \( \{i_n^2\} \) and \( \{e_n^2\} \)  
(b) noise characteristics \( F \) and \( T_e \)  
(c) optimum value of \( R_S \)  
(d) optimum value of source current \( I = I_C \) if \( R_S = 4k\Omega \) (as shown).

Assume \( \beta_F = 100 \), \( r_b = 25\Omega \), and \( V_T \) (same as \( kT/q \)) = .025V.

**ANALYSIS:** For the current shown \( \quad g_m = 40 \times .025mA = 10mA/V \)
For which

\[ N_{OC} = 4kTR_L + \{i_n^2\} \times (R_L')^2 = (1.6 \times 10^{-20}) \times 6k\Omega + (0.8 \times 10^{-22}) \times (6k\Omega)^2 \]

\[ = (9.6 \times 10^{-17}) + (28.8 \times 10^{-16}) = 29.76 \times 10^{-16} V^2 / Hz \]

And therefore the input equivalent noise terms are

\[ \{i_n^2\} = 0.8 \times 10^{-24} A^2/Hz \]

\[ \{e_n^2\} = \left(4kTR_B + N_{OC}/A_v^2\right) = 0.4 \times 10^{-18} + 29.76 \times 10^{-16} / (60)^2 = 1.24 \times 10^{-18} V^2/Hz \]

*Since the exercise involves something of a calculation grind, take note of the systematic process that is followed in order to keep the magnitudes straight. Also take note of the fact that the values for \{i_n^2\} and \{e_n^2\} are similar in magnitude to those for \{i_n\} and \{e_n\} extracted in Exercise E20.4-1.

(b) In like manner, \( 4kTR_S = (1.6 \times 10^{-20}) \times (4 \times 10^3) = 6.4 \times 10^{-17} V^2/Hz \)

\[ F = 1 + \frac{1.24 \times 10^{-18} + (4 \times 10^3)^2 \times (0.8 \times 10^{-24})}{6.4 \times 10^{-17}} = 1 + \frac{0.124 + 1.28}{6.4} = 1.22 \]

For which \( T_e = (1.22 -1) \times 300 = 66K \)

(c) The optimum value of \( R_S \) would have been

\[ R_S(\text{optimum}) = \sqrt{\frac{1.24 \times 10^{-18}}{0.8 \times 10^{-24}}} = 1.25k\Omega \]

(d) The optimum value of \( I_c \) would have been

\[ I \approx \sqrt{\frac{\beta F}{R_S}} \frac{V_T}{R_S} = \sqrt{100 \times \frac{0.025}{4.0k}} = 0.0625mA \]
20.5 NOISE NOMENCLATURE AND ANALYSIS USING SPICE

It should be evident from exercise 20.4-2 that noise analysis can entail an overload of computational overhead. Each resistor and transistor contributes to the noise calculation as either a noise current or a noise voltage generator. The circuit simulation environment can handle this overhead without too much complaint. It does so via the noise-spectral density (NSD), and the user can assess full-spectrum effects using post-processor utilities.

In the SPICE simulation environment, each noise source is assessed one-by-one as a source embedded within the circuit topology, and gives its result in terms of the individual and collective effect at the output node. Note that the output point must therefore be identified. Results in each instance are in terms of the NSD form of $V^2/Hz$, not unlike that entertained in section 20.4.

The specific mathematics of the noise contributions will be associated with the topology. For example the SPICE nomenclature $NTOT(R_s)$ identifies the noise contribution from the resistance $R_s$ at the output, for which

$$NTOT(R_s) = 4k_BT R_s \times (\text{input } V_{\text{div}})^2 \times \left(\frac{v_l}{v_i}\right)^2$$  \hspace{1cm} (20.5-1)

The input voltage divider ($\text{input } V_{\text{div}}$) depends on $R_{in}$ and $R_s$, and the transfer gain $v_l/v_i$ depends on the circuit topology.

For a shot noise source, such as that given by equation (20.4.7a) (= base current noise), SPICE uses the nomenclature $NSIB(Q1)$, for which $Q1$ is the affiliated transistor. The specific mathematics for the term $NSIB(Q1)$ is

$$NSIB(Q1) = (i_{nb}^2 \times R_{BS}^2) \times \left(\frac{\text{base } V_{\text{div}}}{v_{B}}\right)^2 \times \left(\frac{v_l}{v_B}\right)^2$$  \hspace{1cm} (20.5-2)

where the $\text{base } V_{\text{div}}$ is the voltage divider that a source at the base node will encounter and $v_l/v_B$ is the transfer gain to the output node from the base node.

This analysis would be an excess of grind if done by hand, but SPICE will do it all quite handily and with much less trauma.

SPICE will provide a complete breakout of NSD terms in the user’s behalf, with names such as $NTOT(Rs)$, $NSIB(Q1)$, $NRB(Q1)$, each of which are individual contributions to the NSD as read at the output. Preliminary knowledge of the noise effects is necessary to interpret the nomenclature.
The total contribution at the output is given by \( NTOT(ONOISE) \). This is a noise spectral density (NSD) and therefore has amplitude that relates 1:1 to the pass band of the circuit. The total output noise is the sum of all noise contributions over the entire spectrum, and can be accomplished by the pSPICE summing (same as integral) function \( S(\ ) \). Therefore the noise analysis

\[
e_n^2(out) = \int_{0}^{\infty} NTOT(ONOISE) df
\]

will be accomplished by SPICE as \( S(NTOT(ONOISE)) \). Signal/noise ratio (S/N) is then

\[
S \over N = \frac{v_{L}^2}{e_n^2(out)} = \frac{Max(Vout) \cdot Max(Vout)}{Max(S(NTOT(ONOISE)))}
\]

In terms of the output node, the noise factor \( F \) will be

\[
F = \frac{Max(NTOT(ONOISE))}{Max(NTOT(RS))}
\]

for which \( NTOT(RS) \) is the (NSD) noise contribution at the output due to the input source resistance \( R_S \), whereas \( NTOT(ONOISE) \) is the NSD due to everything, including \( R_S \).

Using these postprocessor functions, and the parameter declaration and variation option, noise analysis characteristics such as those of (20.5-4) and (20.5-5) can be evaluated for the effect of circuit design choices such as \( R_S \) and \( I_C \).

**EXAMPLE 20.5-1:** Determine \( R_S(\text{opt}) \) and the associated noise factor \( F \) for the CE circuit topology of figure E20.5-1 using pSPICE.

**ANALYSIS:** Execute a frequency sweep from 50Hz to 50MHz with the noise enabled. Using the PARAMETER call and the Analysis>Setup>Parametric menu, vary the source resistance \( R_S \) from 1k to 10k, call up the goal function menu for the post processor, and plot noise factor \( F \) (given by equation (20.5-5)) vs \( R_S \). Find the value of \( R_S \) for which \( F \) is a minimum.
20.6 DYNAMIC RANGE AND CUBIC DISTORTION

Distortion is primarily a consequence of amplitude and non-linearities in the transfer curve. Most of the more severe distortion manifests itself as a “flattening” of the waveform as result of the signal approaching its compliance limits (associated with the voltage rails). It is also a result of the linear approximations for the active devices (transistors) being exceeded, in which case it will be manifested itself as higher-order harmonics and their Fourier amplitudes.

In the analysis of the power gain, quadratic terms are usually eliminated by means of a differential stage (such as an opamp). That means that the next higher-order terms will be cubic. This distortion is
represented by figure 20.6-1, and can be regarded as if it were a mixing of signals, as shown. Consequently it is also often called *intermodulation distortion*.

![Waveforms and cubic distortion](image)

**Figure 20.6-1:** Waveforms and cubic distortion.

Distortion power for cubic distortion relates to the input power level as

\[ P_d = kP_i^3 \]  

(20.6-1)

A plot of the relative output levels is represented by figure 20.6-2.

![Output power vs cubic distortion power](image)

**Figure 20.6-2:** Output power \( P_o \) vs cubic distortion power as represented by equation (20.6-1)

The figure shows that at some input level of power, \( P_i = P_I \), the power associated with cubic distortion will begin to dominate the output power. \( P_I \) is called the *two-tone intercept level*.

Comparing \( P_d \) to \( P_i \) by means of a ratio gives

\[ \frac{P_d}{P_o} = \frac{kP_i^3}{GP_i} = (K_i P_i)^2 \]  

(20.6-2)

where \( G = \) power gain of the amplifier. And since \( P_d/P_o = 1 \) when \( P_i = P_I \), then \( K_i = 1/P_i \) and
In analogy to noise analysis, distortion may be treated as if it were due to an input $P_{di}$

$$\frac{P_d}{P_o} = \left( \frac{P_i}{P_i} \right)^2$$

If $P_{di}$ is treated as an input at which the signal quality falls below useful levels, it is analogous to that for the noise floor, $N_f$. And then equation (20.6-3) gives

$$\frac{N_f}{P_i} = \left( \frac{P_i}{P_i} \right)^2$$

which corresponds to limit on input power $P_i$ of

$$P_i = \left( P_i^2 N_f \right)^{1/3}$$

The dynamic range was identified in section 20.1 as

$$DR = 10 \log \left( \frac{P_i = \text{distortion limit power}}{\text{noise limit power} = N_f} \right)$$

therefore

$$DR = 10 \log \left( \frac{P_i^2 N_f^{1/3}}{N_f} \right) = 10 \log \left( \frac{P_i}{N_f} \right)^{2/3}$$

and the dynamic range (DR) for cubic distortion reduces to

$$DR = \frac{2}{3} \left( 10 \log P_i - 10 \log N_f \right)$$

And for most cases of signal conditioning for which we are assessing the limits if a system, this is the definition of dynamic range that will be assumed. Note that equation (20.6-5) is oriented toward $P_i$ and $N_f$ being expressed in units of either dBW or dBm.
EXAMPLE 20.6-1: Assume that the system identified by example 20.3-2 has a two-tone intercept point 
\( P_I = -10\text{dBW} \). (a) what is the dynamic range of the system (b) what is the DR of the system if a linear 
noiseless preamplifier of gain 10dB is inserted before the first stage?

SOLUTION: The result of example 20.2-2 was that noise floor \( N_f = -143 \text{dBW} \).

(a) From equation (20.6-5) we then get a DR of

\[
DR = \frac{2}{3} ( -10 \text{dBW} - (-143 \text{dBW}) ) = 88.7 \text{dB}
\]

(b) If a preamplifier of power gain \( G_1 = 10\text{dB} \) is added to the input, distortion will occur for an even lower 
power input level, namely \( P'_I = -20\text{dBW} \). Recalculating the noise factor, and assuming value of \( F_2 = 6.61 \), 
as identified by example 20.2-2 gives

\[
F' = F_1 + \frac{F_2 - 1}{G_1} = 1 + \frac{6.31 - 1}{10} = 1.53
\]

which corresponds to 1.85dB, for which the noise floor will then become \( N_f = -149.2\text{dBW} \)

and the dynamic range will be

\[
DR = \frac{2}{3} ( -20 \text{dBW} - (-149.2 \text{dBW}) ) = 86.1 \text{dB}
\]

Take note that the added amplifier serves to reduce the noise floor, but concurrently will also decrease the 
DR, courtesy of the effects on the distortion, as reflected by the change in the intercept point \( P_I \).
SUMMARY

Noise: \( 4kT = 1.6 \times 10^{-20} \) J

Thermal noise (NSD) \( \{ e_n^2 \} = 4kTR \)
\( \{ i_n^2 \} = 4kTG \)

Shot noise: (NSD) \( \{ i_n^2 \} = 2qI \) for BJT
\( \{ i_n^2 \} = 4kT \times \frac{2}{3} g_m \) for FET
\( \{ e_n^2 \} = \{ i_n^2 \} \times R^2 \)

Noise reflected from output to input
\( e_n^2 (in) = e_n^2 (out) / A_v^2 \)

Noise factor:
\( F = 1 + \frac{N_s}{N_i} = 1 + \frac{T_e}{T} \)

Noise temperature:
\( T_e = T_1 + T_2 / G_1 + T_3 / (G_1 G_2) + \cdots \)

In general:
\( F = 1 + \frac{\{ e_n^2 \} + \{ i_n^2 \} \times R_S^2}{4kTR_S} = \frac{R_s (opt)}{\sqrt{\{ e_n^2 \} / \{ i_n^2 \}}} \)

Noise floor:
\( S_f (\text{min}) = kT \times B \times F \times \left( S / N \right)_{\text{out}} \)
\[ 10 \log kT = -204 \text{ dB} \]

Minimum detectable signal:
\( \text{vs}(\text{min}) = \sqrt{4R_s S_f (\text{min})} \)

Dynamic range:
\( DR = \frac{2}{3} \left( \log(P_i) - \log(N_f) \right) \)