CHAPTER 8. AC POWER AND THE POWER GRID

8.1 AC POWER

Power is the consumption of energy, whether it be by thermodynamics, livestock, or systems. In the electrical world it translates into the product of current and voltage. Inasmuch as AC power involves sinusoidal voltages and currents, product of $V(t)$ and $I(t)$ becomes a little more of an analysis challenge than the steady-state form since these quantities have phase. Since reactive elements in the circuit shift the phase relationship between $I$ and $V$ the amplitude-phase relationship becomes a dominant factor in the environment in which power is transferred and applied.

Consider a current-voltage situation for which the phases are not the same, i.e.

\[
V(t) = V_m \cos (\omega t + \phi_V) \quad (8.1-1a)
\]

\[
I(t) = I_m \cos (\omega t + \phi_I) \quad (8.1-1b)
\]

Then the power absorbed or released is defined by the time average

\[
\langle p(t) \rangle = \frac{1}{T} \int_0^T V_m \cos(\omega t + \phi_V) \times I_m \cos(\omega t + \phi_I) dt
\]

\[
= \frac{1}{2\pi} V_m I_m \int_0^{2\pi} \cos(x + \phi_V) \times \cos(x + \phi_I) dx
\]

\[
= \frac{1}{2} V_m I_m \cos(\phi_V - \phi_I) \quad (8.1-2)
\]

A trigonometric identity and some fun math is involved in deriving equation (8.1-2). Otherwise it is not unlike the definition of power for DC (steady-state) levels except for the time-average amplitude factor $\frac{1}{2}$ and the $\cos()$ of the phase difference. If the current and voltage differ in phase with one another by 90⁰, then no power is dissipated or transferred to the branch in question. Purely reactive components such as capacitance and inductance fall into this category since their currents and voltages are always in quadrature. Consequently

\[
\langle p(t) \rangle = 0 \quad \text{for either } C \text{ or for } L
\]

Capacitance and inductances will store energy but not consume it. They are also called lossless components.

Equation (8.1-2) implies that assessment of AC power must always include a power factor term of the form
\[ \text{pf} \equiv \cos(\phi_V - \phi_I) \quad (8.1-3) \]

A power factor of 1.0 is always true for resistance components since current and voltage are always in phase. This lends the analysis to the amplitude relationship for AC power for which

\[ \langle p(t) \rangle = \frac{1}{2} V_m I_m = \frac{1}{2} V_m \times \frac{V_m}{R} = \frac{1}{2} \frac{V_m^2}{R} \quad (8.1-4a) \]

\[ \langle p(t) \rangle = \frac{1}{2} V_m I_m = \frac{1}{2} \frac{I_m^2 R}{2} \quad (8.1-4b) \]

It is convenient and a common practice to redefine the amplitude of AC signals in terms of their rms averaged amplitudes

\[ V_{\text{RMS}} = V_{AC} = V_m / \sqrt{2} \quad (8.1-5a) \]

\[ I_{\text{RMS}} = I_{AC} = I_m / \sqrt{2} \quad (8.1-5b) \]

so that

\[ \langle p(t) \rangle = I_{AC} \times V_{AC} = I_{AC}^2 R = V_{AC}^2 / R \quad (8.1-5) \]

for which the units of measure are usually so designated, such as 120 VAC.

**EXAMPLE 8.1-1:** The following R-L circuit represents an impedance load which will induce a shift of phase between the voltage and current. The voltage source is 14.1 VAC. Determine:

(a) complex impedance
(b) magnitude and phase of the impedance
(c) current into the impedance
(d) amplitudes \( V_m \) and \( I_m \)

**SOLUTION:** (a) The circuit impedance is \( Z = 40 + j \omega L \)

where \( \omega L = 2\pi \times 60\text{Hz} \times 80\text{mH} \approx 60 \times 500 \text{ m\Omega} = 30 \text{ \Omega} \)

So \( Z = (40 + 30j) \Omega \)

(b) Since it is a complex number the magnitude and phase resolve as
\[ |Z| = \sqrt{40^2 + 30^2} = 50 \, \Omega \quad \phi_Z = \tan^{-1}(30/40) \approx 37^\circ \]

Also written as \( Z = |Z|e^{j\phi_Z} \)

(c) Then \( i = v/Z = V_m/|Z|e^{-j\phi_Z} = 0.4Ae^{-j37^\circ} \) or \( \phi_I = -37^\circ \)

(d) The voltage magnitude is \( V_m = \sqrt{2} \times V_{AC} = 20 \, V \)

The current magnitude is \( I_m = V_m/|Z| = 20/50 = 0.4A \)

### 8.2 PHASOR NOTATION

Section 8.1 and example 8.1-1 acknowledge that the magnitude and the phase are inherent to circuits and are the context necessary for the assessment of AC power as it is distributed within a circuit. The mathematics of sinusoidal current and voltage relationships also appeared in chapter 5 in which the frequency domain reset the analysis in complex form. As represented by example 8.1-1 the magnitude and phase relationship is most cleanly represented by the phasor form and makes the most sense in terms of the relationships involving impedance and admittance. The phasor form is represented mathematically as \( |Z| \angle \phi \) and has a vector graphics representation as indicated by figures 8.2-1 and 8.2-2. These figures, which are for inductance and capacitance circuits, respectively, also suggest the use of impedance triangles as a means to interpret load behavior in the analysis of AC power.

**Figure 8.2-1.** Phasor notation for impedance, RL circuit and impedance triangle.
EXAMPLE 8.2-1: Evaluate the circuits shown and determine:
(a) phasor representation of the impedance
(b) AC current and voltage
(c) power dissipated in the circuit and
(d) power factor ($\text{pf}$)

SOLUTION: (a) The impedance is $Z = 12 \text{k} \Omega + j \omega L$

where $\omega L = 2\pi \times 16\text{MHz} \times 35\mu\text{H} = 100\text{Mr/s} \times 35\mu\text{H} = 3.5 \text{k} \Omega$

and so $Z = 12 + 3.5j$ \[ |Z| = \sqrt{12^2 + 3.5^2} = 12.5 \Omega \quad \phi_Z = \tan^{-1}(3.5/12) \approx 16^\circ \]

(b) The signal voltage is $v_s = 5.0 \angle 0^\circ$ and therefore

\[ i = \frac{v}{Z} = \frac{5.0 \angle 0^\circ}{12.5 \angle 16^\circ} = 0.4 \angle -16^\circ \]

(d) The power factor is $\text{pf} = \cos(-16^\circ) = 0.96$

(c) And therefore the power dissipated in the circuit is $\langle p(t) \rangle = 5.0 \times 0.4 \times 0.96 = 1.92 \text{W}$

Figures 8.2-1 and 8.2-2 are the precursor to a graphical form that allows impedances and admittances to be treated as vector quantities and added head-to-tail. The graphical usage can be extended by mapping...
the complex plane into a set of tangential circular coordinates. For the construct shown, real and imaginary coordinates are orthogonal, just like they are for rectangular coordinates. A circuit construct then becomes a path in this map, devised from its constituent impedances and admittances. Because it is a circular construct the map includes the entire complex plane from zero to infinity. The construct is a quick method of evaluating the magnitude and phase relationships of a linear circuit and is called a Z-Smith chart as represented by figure 8.2-4.

The Complete Smith Chart

Black Magic Design

Figure 8.2-4. Smith chart. It finds its greatest usage in the UHF domain (MHz and GHz) where wavelengths are on the order of cm. For $f = 60\text{Hz}$ the wavelength is 5000km.
8.3 COMPLEX POWER and THE POWER GRID

Even though AC power analysis is a generic science, the primary client is that of the power grid. And since the power grid is essential to rotating machinery, it favors low frequencies. International line voltage standards are predominantly 60Hz/120VAC or 50Hz/240VAC, with some variations from one geopolitical region to the next. The predominant ones are

\[ f = 60\text{Hz} \, @120\text{VAC} \quad \text{for the America, part of Asia, parts of Japan} \]
\[ f = 50\text{Hz} \, @240\text{VAC} \quad \text{for Europe, most of the rest of the world} \]

The choice of frequency for the power grid is somewhat arbitrary, almost as is the definition of units of time, which are based on the hour and then subdivided into minutes and then further subdivided into seconds). The ancient Egyptians and Greeks gave us the 24-hour day. The two later levels of subdivision by 60 (minutes), and then 60 (seconds) had better foresight since the number 60 has factors 1, 2, 3, 4, 5, and 6 (as well as 10, 12, 15, 20). And since the number 60 has these properties and since it has an analogy to time units it has become favored as the grid frequency for many geopolitical regions.

There is an entertaining story about the ‘Current Wars ’ over the question of DC vs AC power grids in the late 19th century, with the names of such notables as Edison, Westinghouse and Tesla involved. The AC power grid prevailed, courtesy of the transmission efficiency afforded by the step-up-step-down transformer (previous chapter) and as a consequence of a number of improvements in transformer core materials. HVDC grids are now coming back into play because of the need to interface different sectors and energy conversion technologies that favor both DC and AC.

As an aside, the rotational frequency for a rotor in RPM is given by

\[ N_R = \frac{120f}{n_p} \quad (8.3-1) \]

where \( n_p \) is the number of magnet poles set into the rotor. A large 12-pole rotor and \( f = 60\text{Hz} \) would have a rotational speed of 600 RPM. At higher rotational speeds, centripetal effects on a large rotor would be catastrophic. Equation (8.3-1) is subject to modification if the stator coils are interlaced.

The radian measure of the two primary grid frequencies and amplitude values of the AC voltages are

\[ \omega = 377 \text{ r/s for } f = 60\text{Hz} \quad V_m = 170\text{V for VAC = 120v} \]
\[ \omega = 314 \text{ r/s for } f = 50\text{Hz} \quad V_m = 340\text{V for VAC = 240v} \]

and these will be used as defaults for examination of the power grid.

Even with the onset of new sources of power, the power grid environment is still primarily vested in the sinusoidal forms. In order to accommodate the various industrial and residential loads, the current,
voltage, impedance, and admittance are best served when they are represented by phasors. Courtesy of the fact that AC voltage and current levels are commonly represented by rms levels it is appropriate to make the distinction

\[ V_m = \sqrt{2} \left| v(\omega) \right| \quad I_m = \sqrt{2} \left| i(\omega) \right| \]

for which the voltages and currents will be

\[ V(\omega) = V_m \angle \phi_v \] \hspace{1cm} (8.3-2a)
\[ I(\omega) = I_m \angle \phi_i \] \hspace{1cm} (8.3-2b)

It is then of practical value to define \( S = v \times i^* \) as the apparent power for which the phasor forms is

\[ S = v \times i^* = \frac{1}{2} (V_m \angle \phi_v)(I_m \angle \phi_i)^* = \frac{1}{2} V_m I_m \angle (\phi_v - \phi_i) \] \hspace{1cm} (8.3-3)

(the asterisk means ‘complex conjugate’) (which is another way of saying ‘switch the sign on the imaginary component’). It also reconfirms that the phase difference between \( V \) and \( I \) is of essential to the context of AC power, consistent with equation (8.1-2).

The resulting phasor form is also a complex number, which may be represented in either phasor form or in terms of real and imaginary components, i.e.

\[ S = \frac{1}{2} V_m I_m \left[ \cos(\phi_v - \phi_i) + j \sin(\phi_v - \phi_i) \right] \] \hspace{1cm} (8.3-4)

\[ = P + jQ \] \hspace{1cm} (8.3-5)

for which

\[ P = \frac{1}{2} V_m I_m \cos(\phi_v - \phi_i) = V_{AC} I_{AC} \cos(\phi_v - \phi_i) \] \hspace{1cm} (8.3-6a)
\[ Q = \frac{1}{2} V_m I_m \sin(\phi_v - \phi_i) = V_{AC} I_{AC} \sin(\phi_v - \phi_i) \] \hspace{1cm} (8.3-6b)

Equation (8.3-6a) is called the average power and is the same as equation (8.1-2). Equation (8.3-6b) is called the reactive power. Reactive power \( Q \) does nothing since it is out of phase with the real power, but it plays a critical role in adjustment of the power factor.

The phasor usage also fits the determination and definition of impedance since
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\[ Z = \frac{V_m \angle \phi_v}{I_m \angle \phi_i} = \frac{V_m}{I_m} \angle (\phi_v - \phi_i) = |Z| \angle (\phi_v - \phi_i) \]  

(8.3-7)

The apparent power also relates to impedance \( Z \) as

\[ S = \frac{1}{2} I_m V_m \angle (\phi_v - \phi_i) = \frac{1}{2} I_m \left| Z \right| \angle (\phi_v - \phi_i) = \frac{1}{2} I_m^2 Z \]  

(8.3-8)

for which the real and imaginary parts are then

\[ P = \frac{1}{2} I_m^2 \text{Re}[Z] \quad Q = \frac{1}{2} I_m^2 \text{Im}[Z] \]  

(8.3-9)

consistent with equations (8.3-6a) and (8.3-6b). The nomenclature of power grid engineering is then

\[ P \equiv \text{average power: } (\text{This is what does the work}) = \frac{1}{2} I_m V_m \cos(\phi) \text{. Measure in Watts.} \]

\[ Q \equiv \text{reactive power: } = \frac{1}{2} I_m V_m \sin(\phi) \text{. Measure in VAR} \]

\[ S \equiv \text{apparent power: } = \frac{1}{2} I_m V_m = I_{AC} V_{AC} = I_{rms} V_{rms} = I_{eff} V_{eff} \text{. Measure in VA} \]

The AC amplitudes \( V_{AC} \) and \( I_{AC} \) are also applicable to \( P \) and \( Q \).

In addition to these definitions the power factor \( pf \) is also a specification, defined either by the phase difference between \( v \) and \( i \) or by their effect on the average power i.e.

\[ pf \equiv \cos(\phi_v - \phi_i) \]  

(8.1-3)

With the power grid the \( pf \) is also cited as either \textit{leading} or \textit{lagging}, which is the attitude of the AC current relative to the AC voltage. It is also an indication of the type of load, i.e.

\begin{itemize}
  \item \textit{lagging} \quad \text{(i lags v)} \quad \text{inductive load}
  \item \textit{leading} \quad \text{(i lead v)} \quad \text{capacitative load}
\end{itemize}

\textit{(pf specifications)}:

For convenience, and because of the relationship of power to impedance, the power specifications are also indicated in terms of a \textit{power triangle}, as shown by figure 8.3-1
EXAMPLE 8.3-1: Typical specifications for the power grid/utility loads are

1. 50 kVA load @ pf = 0.8 lagging
2. 50 kW load @ pf = 0.9 lagging
3. 20 kVAR load @ pf = 0.7 leading

Identify the elements of the apparent, real and reactive load elements, \((S,P,Q)\) for each case.

SOLUTION:

1. \(|S| = 50 \text{kVA (specified)}\) \quad \therefore P = 0.8 \times 50 = 40 \text{kW}
   
   and \(Q = \sqrt{50^2 - 40^2} = 30 \text{kVAR}\)

2. \(P = 50 \text{kW (specified)}\) \quad \therefore |S| = 50/0.9 = 55.6 \text{kVA}
   
   \(\phi = \cos^{-1}(0.9) = 25.8^\circ\)
   
   \(Q = 55.6 \sin(25.8^\circ) = 24 \text{kVAR}\)
   
   And also \(Q = \sqrt{55.6^2 - 50^2} \approx \sqrt{586} \approx 24 \text{kVAR}\)

3. \(Q = -20 \text{kVAR (specified)}\) (the negative sign is used because it is a leading phase)
   
   \(\phi = \cos^{-1}(0.7) = -45^\circ\) \quad \text{(negative sign because it is a leading phase)}
   
   \(P = 20/\tan 45 = 20 \text{ kW}\)
   
   \(S = \sqrt{20^2 + 20^2} = 28.3 \text{kVA}\)
There is a shortcut context to example 8.3-1 that is suggested. And this context is represented by figure 8.3-2.

**Figure 8.3-2. Angles and power factors**

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**EXAMPLE 8.3-2: Example application.**

Two loads are represented. Find $P$, $Q$, $I_A$, $I_B$, $I_{rms}$. And indicate the power triangle for each load and the composite power triangle for the two loads.

**SOLUTION:**

For **load A**: the apparent power $|S_A| = 10$ kVA (given)

And, for $pf = 0.5$ then $P_A = |S| \times pf = 10 \times 0.5 = 5$ kW

Since the load is leading then the phase angle $\phi_A$ is negative and so $Q_A$ is also negative

$$Q_A = -\sqrt{S_A^2 - P_A^2} = -\sqrt{10^2 - 5^2} = -\sqrt{75} = -8.66 \text{ kVAR}$$

And $S_A$ has phase angle $\phi_A = \cos^{-1}(0.5) = -60^\circ$ (= negative since leading). The power triangle (shown) is a graphical representation, and is as good if not better than the mathematics.

For **load B**: the average power $P_B = 5$ kW is given.

For $pf = 0.7$ lagging, the angle $\phi_B$ is positive, and via figure 8.3-2, $\phi = +45^\circ$

So $Q_B = +P_B \tan 45^\circ = 5 \times 1.0 = 5.0 \text{ kVAR}$

And $|S_B| = \sqrt{P_B^2 + Q_B^2} = \sqrt{5^2 + 5^2} = 5\sqrt{2} = 7.1 \text{ kVA}$

The power triangle is as shown.
Adding the real component magnitudes (horizontal vectors) gives composite real power:

\[ P = P_A + P_B = 5.0 \text{ kW} + 5.0 \text{ kW} = 10 \text{ kW} \]

Adding the reactive component composite vector is:

\[ Q = Q_A + Q_B = -8.66 + 5.0 = -3.66 \text{ kVAR} \]

So the apparent power magnitude is

\[ |S| = \sqrt{P^2 + Q^2} = \sqrt{10^2 + 3.66^2} = 10.6 \text{ kVA} \]

\[ \phi = \tan^{-1}(Q/P) = \tan^{-1}(-3.66/10) = -19.6^\circ \]

And so \[ S = 10.6 \angle -19.6^\circ \] is the result.

The power factor is either

\[ pf = \cos(-19.6) = 0.942 \]

or

\[ pf = P / |S| = 10/10.6 = 0.94 \]

The power triangles show the result. Notice that the mathematics could have been accomplished graphically or could have been used for WAG (wild-eyed guess) solution values.

The currents are then derived by:

\[ I_A = S / V_S \angle \phi_V = \frac{10 \text{ kVA} \angle -60^\circ}{1 \text{ kV} \angle 30^\circ} = 10 \angle -90^\circ \text{ A} \]

So \[ I_A = (10 \angle -90^\circ)^* = 10 \angle +90^\circ \text{ A} \]

\[ I_B = S / V_S \angle \phi_V = \frac{7.1 \text{ kVA} \angle 45^\circ}{1 \text{ kV} \angle 30^\circ} = 7.1 \angle 15^\circ \text{ A} \]

So \[ I_B = (7.1 \angle 15^\circ)^* = 7.1 \angle -15^\circ \text{ A} \]

\[ I_{rms} = |S| / V_{rms} = (I_{rms} \ V_{rms}) / V_{rms} = (10.6 \text{ kVA}) / 1 \text{ kV} = 10.6 \text{ A} \]

Although it is not quite so critical with the previous example the power factor \( pf \) is the benchmark of power grid load considerations, particularly those for which there is heavy machinery. The load will then be inductive, for which the \( pf \) will be lagging as reflected by page the statement at the bottom of page 160.

And therefore the \( pf \) is adjusted by adjustment of the imaginary component \( Q \) of the AC power, usually means of a capacitor bank. Consider the following example.
EXAMPLE 8.3-3: A 400kW load with $pf = 0.8$ lagging is supplied by a 1.0 kVAC source. What capacitance is required to change the $pf$ to $= 0.95$ lagging. Assume $f = 60$Hz.

**SOLUTION:** For the load as specified $P = 400$ kW,

∴ $|S_1| = (400kW)/0.8 = 500$ kVA and $Q_1 = \sqrt{S_1^2 - P^2} = \sqrt{500^2 - 400^2} = 300$ kVAR

For the load as desired, $|S_2| = (400kW)/0.95 \approx 421$ kVA and $Q_2 = \sqrt{421^2 - 400^2} = 131.5$ kVAR

so in order to achieve the desired power factor $\Delta Q = (131.5 - 3.0)$ kVAR = - 161.5 kVAR must be added. The negative sign indicates that the change in $Q$ must be achieved by a capacitance load.

Since (reactive) power $Q = V^2/Z = YV^2$, then $Y_C = \Delta Q/V^2$

And $\omega C = 161.5$ kVAR/(1.0kV)$^2$

or $C = (161.5 \times 10^{-3})/(2\pi \times 60$Hz) = 431 \mu F

So a 431 \mu F capacitance with voltage rating $V = \sqrt{2} \times 1.0$ kVAC = 1.414 kV is needed to achieve a 0.95 power factor for this load.

The power triangle indicates the graphical interpretation.

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### 8.4 THREE-PHASE POWER

Since the power grid must support both industrial and residential environments, some innovation is required in the deployment. Three-phase power is a more efficient and smoother form for supporting ‘rotor and motor’ applications than is single phase. Three-phase motors are smaller and more compact than their single-phase cousins. The total 3-phase power supplied to a balanced 3-phase is constant at every instant of time and 3-phase motors have an absolutely uniform torque.

The three-phase waveform is little more than three supply lines (a.k.a. bus lines) at 120° phase difference from one another, represented by figure 8.4-1. Consequently, single-phase residential power is derived from the three-phase supply lines unless you have a residential situation defined by massive machines.
Three-phase systems are usually generated at the power plant with a rotor-stator form similar to that represented by figure 8.4-2. The equivalent schematic of the three-phase source, whether from a power plant or from a portable or from a transformer, is represented by figure 8.4-3.

The neutral line carries no current since the phase difference causes all of the lines to add to zero at the neutral center point. Unless there is a hidden fourth line, high-voltage AC transmission line towers usually only have three lines, $L_1$, $L_2$, $L_3$. 

Figure 8.4-1. Three-phase wave output.

Figure 8.4-2. Three-phase generator. A power plant generator will have many more rotor and stator poles but will still be deployed as three feed lines.
As a matter of nomenclature, the three outer nodes of figure 8.4-3 are called the line terminals and connect to the (transmission) lines. Consequently the current flowing in the lines are called the line currents. The voltages of the outer nodes relative to the center (neutral) node are called the phase voltages. The voltage between any two of the outer nodes, also between lines, are called the line voltages.

The introduction of the symmetrical topology also reflects the fact that either load or source impedances can be configured as one of two options, affectionately known as the ‘Wye’ and the ‘Delta’ forms. They are represented by figure 8.4-4.

If we reference the figure and compare, the resistance between nodes N1 and N3 would be

\[
R_1 + R_3 = R_b \left( R_a + R_c \right) = \frac{R_b \left( R_a + R_c \right)}{R_a + R_b + R_c} \quad (8.4-1a)
\]

The resistance between nodes N2 and N3 would be
\[ R_2 + R_3 = R_C \| (R_A + R_B) \] (8.4-1b)

The resistance between nodes N1 and N2 would be

\[ R_1 + R_2 = R_A (R_C + R_B) \] (8.4-1c)

If equation (8.4-1b) is subtracted from (8.4-1a), common terms will subtract out and give

\[ R_1 - R_2 = \frac{R_B R_A - R_C R_A}{R_A + R_B + R_C} \] (8.4-1d)

And if Equations (8.4-1c) and (8.4-1d) are added then

\[ R_1 = \frac{R_B R_A}{R_A + R_B + R_C} \] (8.4-2a)

Equation (8.4-2a) nicely separates \( R_1 \) of the Y network of figure (a) and defines it in terms of the impedances of the \( \Delta \) network of figure (b). If the cycle is carried through the other resistance subscripts, et. seq. then

\[ R_2 = \frac{R_A R_C}{R_A + R_B + R_C} \] (8.4-2b)

\[ R_3 = \frac{R_C R_B}{R_A + R_B + R_C} \] (8.4-2c)

Equations (8.4-2a), (8.4-2b) and (8.4-2c) are called a Delta-to-Wye conversion.

The Wye-to-Delta conversion takes a good bit more fun mathematics than the Delta-to-Wye conversion. If we do product multiplication and sum of equations (8.4-2a), (8.4-2b) and (8.4-2c) then

\[ R_1 R_2 + R_2 R_3 + R_3 R_1 = \frac{R_A R_B R_C (R_A + R_B + R_C)}{(R_A + R_B + R_C)^2} = \frac{R_A R_B R_C}{R_A + R_B + R_C} \] (8.4-3)

Dividing equation (8.4-3) by equation (8.5-2c) gives

\[ R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \] (8.4-4a)
Similarly
\[ R_B = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \]  
(8.4-4b)
\[ R_C = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \]  
(8.4-4c)

Equations (8.4-4a), (8.4-4b) and (8.4-4c) are called a Wye-to-Delta conversion.

The Y-Δ conversion equation sets can and usually are stated in terms of impedances, \( Z_1, Z_2, Z_3 \) and \( Z_A, Z_B, Z_C \), particularly for the AC power grid.

With the AC power grid it is also common to make the loads balanced, in which case
\[ Z_1 = Z_2 = Z_3 = \frac{Z_Y}{3} \]  
(8.4-5a)
\[ Z_A = Z_B = Z_C = \frac{Z_\Delta}{3} \]  
(8.4-5b)

For which the conversion equations then gratuitously simplify to
\[ Z_Y = \frac{Z_\Delta}{3} \]
\[ Z_\Delta = 3Z_Y \]  
(8.4-6)

Consider the following example

**EXAMPLE 8.4-1:** Example application. The output of a three-phase transformer has phase voltage \( V_P \) = 120VAC at three phases 120° apart as shown by figure E8.4-1(a). The system is balanced, with line impedance \( Z_L = (2.0 + 6j) \Omega \). The load is configured in a delta topology with \( Z_\Delta = (12 + 6j) \Omega \).

Determine the line current \( I_L \) for each phase.

**SOLUTION:** The equivalent \( Z_Y \) to the load impedances of figure E8.5-1(a) is
\[ Z_Y = \frac{Z_\Delta}{3} = \frac{(12 + 6j)}{3} = (4 + 2j) \Omega. \]
The equivalent topology is shown by figure E8.4-1(b)

Each of the Y–Y loops are separate and equal. Choosing loop n-a-A-n’ and evaluating the loop impedance gives

\[ Z = Z_L + Z_Y = (2.0 + 6j) + (4 + 2j) = 6 + 8j = 10 \angle 53^\circ \text{W}. \]

Then the line current is \( I_L = (120 \angle 0^\circ) / (10 \angle 53^\circ) = 12 \angle -53^\circ \text{A}. \)

For each line, the line current will be shifted by 120° relative to this value.

In example 8.5-1 the simplest and most obvious conversion was made. Effort was also made with the topology figure to preserve the context of 3-phases at 120° phase difference, although it would have been simpler to just use Manhattan rules (horizontal-vertical) in the representation of each line loop.

The reason for retaining the Y topology form is represented by figure 8.4-5

Aside from the fact that \( |V_{ac}| \) ‘looks like’ \( \sqrt{3} \times V_P \) the difference

\[ V_{ac} = V_P \angle 0^\circ - V_P \angle -120^\circ = V_P - \left( -0.5 - j\sqrt{3}/2 \right) V_P = \left( 1.5 + j\sqrt{3}/2 \right) V_P \]
proves that it is true since the magnitude of $|V_{ac}| = \sqrt{(9/4) + (3/4)} = \sqrt{3} \times V_p$.

It also has phasor form $V_{ac} = \sqrt{3}V_p \angle 30^\circ$.

Note that for $V_p = 120V$, the line voltage $|V_{ac}| = \sqrt{3} \times V_p = \sqrt{3} \times 120 = 208V$.

Table 8.4-1 shows a summary of the 3-phase connection options and magnitudes.

**Table 8.4-1.** Summary of phase and line voltages and currents for balanced 3Φ systems

<table>
<thead>
<tr>
<th>Connection</th>
<th>Phase voltage/current</th>
<th>Line voltage/current</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y-Y</td>
<td>$</td>
<td>V_{an}</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>I_{a}</td>
</tr>
<tr>
<td>Y-Δ</td>
<td>$</td>
<td>V_{an}</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>I_{a}</td>
</tr>
<tr>
<td>Δ-Δ</td>
<td>$</td>
<td>V_{ab}</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>V_{ab}</td>
</tr>
<tr>
<td>Δ-Y</td>
<td>$V_{ab} = V_p$</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>I_{ab}</td>
</tr>
</tbody>
</table>
PORTFOLIO and SUMMARY

**AC power:** Time averaged \( \langle p(t) \rangle = \frac{1}{2} V_m I_m \cos(\phi_v - \phi_i) \) power factor \( pf = \cos(\phi_v - \phi_i) \)

\[
V_{AC} = V_{RMS} = \frac{V_m}{\sqrt{2}} \quad I_{AC} = I_{RMS} = \frac{I_m}{\sqrt{2}}
\]

\[
\langle p(t) \rangle = I_{AC} \times V_{AC} = I_{AC}^2 R = V_{AC}^2 / R \quad \text{(dissipative load)}
\]

\[
\langle p(t) \rangle = 0 \quad \text{for } C \text{ or } L
\]

**Phasor notation**

Complex power and the power grid

\[ S = P + jQ \]

Apparent power \( S = V \times I^* = \frac{1}{2} (V_m \angle \phi_v)(I_m \angle \phi_i)^* = \frac{1}{2} V_m I_m \angle (\phi_v - \phi_i) \) in VA

Real power \( P = \frac{1}{2} V_m I_m \cos(\phi_v - \phi_i) = V_{AC} I_{AC} \cos(\phi_v - \phi_i) \) in Watts

Reactive power \( Q = \frac{1}{2} V_m I_m \sin(\phi_v - \phi_i) = V_{AC} I_{AC} \sin(\phi_v - \phi_i) \) in VAR

pf specifications: lagging \( (i \text{ lags } v) \) inductive load

leading \( (i \text{ lead } v) \) capacititative load

**Computational shortcuts**

<table>
<thead>
<tr>
<th>( f )</th>
<th>( \omega )</th>
<th>( \text{VAC} )</th>
<th>( V_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60Hz</td>
<td>377 r/s</td>
<td>120v</td>
<td>170V</td>
</tr>
<tr>
<td>50Hz</td>
<td>314 r/s</td>
<td>208v</td>
<td>294V</td>
</tr>
<tr>
<td>50Hz</td>
<td>314 r/s</td>
<td>240v</td>
<td>340V</td>
</tr>
</tbody>
</table>

**Load power triangle**
Three-phase power. Three sinusoidal waves 120° phase differences

The only one we really use the transformation is for the balanced form for which

\[ Z_y = Z_2 = Z_3 = Z' \]
\[ Z_\Delta = Z_\alpha = Z_\epsilon = Z_\Delta \]

for which

\[ Z_y = \frac{Z_\Delta}{3} \]
\[ Z_\Delta = 3Z_y \]

Node difference voltage.

\[ |V_{\Delta}| = \sqrt{3} \times V_p \]

For \( V_p = 120VAC \) \( \rightarrow |V_{\Delta}| = 208VAC \)

Summary table for three-phase power grid topologies

<table>
<thead>
<tr>
<th>Connection</th>
<th>Phase voltage/current</th>
<th>Line voltage/current</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y-Y</td>
<td>( V_{ab} = V_p )</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>I_a</td>
</tr>
<tr>
<td>Y-\Delta</td>
<td>( V_{ab} = V_p )</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>I_a</td>
</tr>
<tr>
<td>\Delta-\Delta</td>
<td>( V_{ab} = V_p )</td>
<td>( V_{ab} = -V_p )</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>I_{ab}</td>
</tr>
<tr>
<td>\Delta-\gamma</td>
<td>( V_{ab} = V_p )</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>I_{ab}</td>
</tr>
</tbody>
</table>